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A Study On Coefficient Bounds for a Newly Defined Subfamily of Convex Functions

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Abstract:
One of the most crucial problems in geometric function theory is the study of the Hankel determinant generated by the Maclaurin series of analytic functions that belong to particular classes of normalized univalent functions. Our goal in this study is first to define a family of convex functions associated with Zigzag coefficients and then to investigate bounds of initial coefficients, Fekete-Szegö inequality, second and third-order Hankel determinants. Further, we also examine the logarithmic coefficients of functions within a defined family regarding recent issues.
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Introduction and Preliminaries

Coefficient problems constitute a fundamental aspect of complex analysis where they serve as key tools for investigating both analytic and univalent functions. Examining these functions requires an understanding of their coefficients to gain essential insights into their behavior growth patterns and fundamental properties. Examine an analytic function $h(z)$ which is defined within the unit disk $U = \{z: |z| = 1\}$and possesses a power series expansion:

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j \qquad (1)$$

We say that the function $h(z)$ is univalent in$U$, if

$$z_1 - z_2 \neq 0 \Rightarrow h(z_1) \neq h(z_2), where\ z_1, z_2 \in U.$$

The collection of all such functions is represented by the symbol S. An analytic function $p \in P$ in U, with $p(0) = 1$ and $Re\big(p(z)\big) > 0$ and can be expressed as follows:

$$p(z) = 1 + \sum_{j=1}^{\infty} c z^j \qquad (2)$$

A function $w(z)$ defined in U said to be a Schwarz function if

$$w(0) = 0\ and\ |w(z)| < 1\ where\ z \in U.$$

The principle of subordination is an essential tool for studying the behavior of various subclasses of univalent functions. Lindelof [1] developed the idea of subordination. In addition, Rogosinki [2,3] and Littlewood [4] conducted an in-depth investigation of it. We say that analytic functions h_1 and h_2 are subordinated denoted by $h_1 < h_2$ if there exists a Schwarz function w such that

$$h_2(z) = h_1\big(w(z)\big). \qquad (3)$$

In particular,

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0)\ and\ h_1(u) \subset h_2(u).$$

The class of starlike functions is represented by $S^*$and is defined as follows:

$$S^* = \left\{h: h \in A\ and\ Re\left(\frac{zh^{'}(z)}{h(z)} > 0\right), z \in U\right\} \qquad (3^*)$$

Similarly, the class of convex functions in U is represented by C and id defined as follows:

$$C = \left\{h: h \in A\ and\ Re\left(1 + \frac{zh''(z)}{h'(z)} > 0\right), z \in U\right\}. \qquad (4)$$

In term of subordinations, these two classes can be written as follows:

$$S^* = \left\{h: h \in A\ and\ \frac{zh^{'}(z)}{h(z)} \prec \frac{1+z}{1-z}, z \in U\right\}. \qquad C =$$
$$\left\{h: h \in A\ and\ \left(1 + \frac{zh''(z)}{h'(z)}\right) \prec \frac{1+z}{1-z}, z \in U\right\}$$

In 1992, the scholars [5] considered a univalent function $\emptyset$ in U with the properties that $\emptyset^{'}(0) > 0$ and $Re\emptyset > 0$ Also, the region $\emptyset(u)$ is star-shaped around the fixed point $\emptyset(0) = 1$ and is symmetric along the real line axis. They proposed the following extensive class that contains more well-known classes in particular cases:

$$S^*(\emptyset) = \left\{h: h \in A \; and \; \frac{zh'(z)}{h(z)} \prec \emptyset(z)\,, z \in U\right\},$$

and

$$C(\emptyset) = \left\{h: h \in A \; and \; \left(1 + \frac{zh''(z)}{h'(z)}\right) \prec \emptyset(z), z \in U\right\}.$$

The logarithmic function coefficients β for the univalent function are defined by using the following series expansion

$$log\left(\frac{h(z)}{z}\right) = 2\sum_{j=1}^{\infty} \beta_j(h) z^j, (z \in U). \qquad (6)$$

For brevity, we use β_j instead of $\beta_j(h)$ These coefficients play a key role for various estimations in the theory of univalent functions. Brennan's hypothesis for conformal mappings was resolved by Kayumov [6] by analyzing the logarithmic coefficients. The Lebedev-Milin inequalities [7]; see also [8,9] demonstrate the importance of the logarithmic coefficients by using them to establish bounds on the coefficients of h Milin [7] conjectured the following inequality:

$$\sum_{m=1}^{i} \sum_{k=1}^{m} \left(k\left|\beta_j\right|^2 - \frac{1}{k}\right) \le 0, (\forall\, j \ge 1)\,.$$

This statement indicates the connection between Robertson's hypothesis [10] and Bieberbach's conjecture [11] ، which is a famous problem concerning the coefficients in the theory of univalent functions. De Branges [12] demonstrated the validity of Bieberbach's conjecture by demonstrating Milin's conjecture:

$$\sum_{j=1}^{m} j(m-j+1)\left|\beta_j\right|^2 \le \sum_{i=1}^{m} \frac{m-j+1}{i} \quad (\forall\, k \ge 1),$$

in which the equality will hold true if h is of the following form:

$$\frac{z}{(1-e^{j\theta}z)^2}$$

for some $\theta \in R$. We can rewrite (6) in the power series form as follows:

$$\log\left(\frac{h(z)}{z}\right) = 2\sum_{j-1}^{\infty} \beta_j(h) z^j = (d_2 z + d_3 z^2 + d_4 z^3 + \cdots)$$
$$-\frac{1}{2}\left(d_2 z + d_3 z^2 + d_4 z^3 + \cdots\right)^2 + \frac{1}{3}\left(d_2 z + d_3 z^2 + d_4 z^3 + \cdots\right)^3$$
$$-\frac{1}{4}\left(d_2 z + d_3 z^2 + d_4 z^3 + \cdots\right)^4 + \cdots$$

Upon equating the coefficients of z^j for $j = 1,2,3$ ... it follows from the above results that

$$\beta_1 = \frac{1}{2} d_2 \qquad (7)$$

$$\beta_2 = \frac{1}{2}\left(d_3 - \frac{1}{2} {d_2}^2\right) \qquad (8)$$

$$\beta_3 = \frac{1}{2}\left(d_4 - d_2 d_3 + \frac{1}{3} {d_2}^3\right) \qquad (9)$$

$$\beta_4 = \frac{1}{2}\left(d_{5-} d_2 d_4 + d^2{}_2 d_3 - \frac{1}{2} {d_3}^2 - \frac{1}{4} {d_2}^4\right) \qquad (10)$$

Recently, a number of authors have conducted researches on problems with the logarithmic coefficients in relation to different classes of univalent functions (see, for example [13-17]) . However, the upper bounds for absolute value of $\beta_j (j \ge 3)$ for univalent functions and for some subfamilies remain unknown. The function $h(z)$ given in (1) has an inverse $\left(h^{-1}\right)$ that is analytic in the disk $|w| < \frac{1}{4}$ If $h \in S$ and if $|w| < \frac{1}{4}$ than

$$h^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots \qquad (11)$$

where

$$A_2 = -d_2 \qquad (12)$$

$$A_3 = 2{d_2}^2 - d_3 \qquad (13)$$

It was shown by Lowner [18] that, if $h \in S$ and its inverse h is provided by (11) , then the following sharp estimate

$$|A_j| \le \frac{(2j)!}{j!(j+1)!}, \qquad (14)$$

Holds true. For any $|A_j|$ $(j = 2,3,4, \ldots)$ in (14), it is shown that the inverse of the Koebe function $K(z) = \frac{z}{(1-z)^2}$ gives the best bounds. One of the main areas of study in geometric function theory has involved determining the upper bounds for the coefficients, as this provides information about many features of the function. For certain subclasses of univalent functions, we are interested in determining the $sup|d_j|$, where $j = 2,3,4, \ldots$ specifically, the corresponding growth and distortion theorems are given by the bound for the second coefficient. The coefficient problems associated with the Hankel determinants are another example. It was also shown by Cantor [19] that, for the quotient of two bounded analytic functions in U, the resulting function is rational, by making use of the Hankel determinants. The Hankel determinant for $h \in S$ given by Pommerenke [20,21] is defined as (15). The j-th Henkel determinant is defined as follows:

$$D_{j,k}(h) = \begin{vmatrix} d_k & \cdots & d_{k+j-1} \\ \vdots & \ddots & \vdots \\ d_{k+j-1} & \cdots & d_{k+2j-2} \end{vmatrix} \qquad (15)$$

where $k, j \in N$, and $d_1 = 1$.

This determinant is a significant item that helps in describing the characteristics of *the associate* analytic functions. The span of applications of the Henkel determinants has been seen in various technological studies, particularly those where mathematical tools are used to a large extent. For example, they are used in the theory of Markov processes and we see their applications in the solutions of non-stationary signals in the Hamburger moment problem. Various subfamilies of univalent functions have had their $D_{j,i}(h)$ development studied. The sharp estimate of this determinant for the family of close- to-convex functions is yet unknown (see [22]) , although Janteng et al. [23,24] discovered the absolute sharp bounds of the functional $D_{2,2}(h)$ for each of the sets C, S^*, and R. However, Krishna et al. [25] demonstrated the best estimate of $D_{2,2}(h)$ for the collection of Bazilevic functions. For a function h of the from (1) the $D_{3,1}(h)$ determinant is defined as follows:

$$D_{3,1}(h) = \begin{vmatrix} 1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{vmatrix} \qquad (16)$$

For the details of the first two cases (see [10,26]) , while Babalola [27] investigated $D_{3,1}(h)$ for the classes C , starlike S^* and bounded turning R. In 2017, Zaprawa [28] improved the results of Babalola and proved that

$$D_{3,1}(h) = \begin{Bmatrix} 1 \quad (h \in S^*) \\ \frac{49}{540} \quad (h \in C) \\ \frac{41}{60} \quad (h \in R) \end{Bmatrix}$$

Recently,Kowalczyk et al. [29] and Lecko et al. [30] obtained sharp bounds for the third order Hankel determinant $|D_{3,1}(h)|$ as follows:

$$|\ D_{3,1}(h)| \le \begin{Bmatrix} \frac{4}{135} \quad (h \in K) \\ \frac{1}{9} \left(h \in S^*\left(\frac{1}{2}\right)\right) \end{Bmatrix}$$

Notably, researchers have explored bounds for the fourth -order Hankel determinant in various analytic function subclasses, including works by Arif et al. [31,32], and wang et al. [33]. For further research in this area, interested readers are referred to [34-41] and the references therein.

Using the technique of subordination, we establish a new class of convex functions connected with $\varphi(z) = \frac{1+z}{cosz}$ as follows:

Definition:1. *An analytic function of the form (1) is said to be in the class C_{ZN}*

$$\frac{\left(zh'(z)\right)'}{h'(z)} = 1 + \frac{h''(z)}{h'(z)} \prec \varphi(z) \quad (z \in U)\ , \qquad (17)$$

Where $$\varphi(z) = \frac{1+z}{cosz} \qquad (18)$$

Set of Lemmas

We shall examine the coefficient problems using the following set of lemmas.

Lemma:1. *(see [42]). Let $p(z) \in P$ .Then*

$$|c_i| \le 2\ , i \in N \qquad (19)$$

Also $|c_i - \mu c_{i-l}| \le 2, i > j, \mu \in [0,1].$ *(20)* *Sharpness holds true for the function*

$h(z)$ given by

$$h(z) = \frac{1+z}{1-z}.$$

Lemma: 2. *(see [32]). Let $p \in P$ Then*

$$\left|c_{i+j} - vc_i c_j\right| \le 2, for\ 0 \le v \le 1. \qquad (21)$$

$$\left|c_{i+2j} - vc_i c_j^2\right| \le 2(1+2v), for\ v \in R. \qquad (22)$$

And

$$\left|c_2 - vc_1^2\right| \le 2max\{1,\ |2v-1|\}\ for\ v \in C. \qquad (23)$$

Lemma: 3. *(See [32]). Let $p \in P$ Then*

$$\left|c_3 - 2Tc_1c_2 + Dc_1^3\right| \le 2,$$

If $0 \le T \le 1, and\ T(2T-1) \le D \le T.$

Lemma: 3. *(See [43] and [42]). If $p \in P$ then*

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

And

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - (4-c_1^2)c_1x^2 + 2(4-c_1^2)(1-|x^2|)z,$$

Where $x, z \in C$ with $|z| \le 1$ and $|x| \le 1$.

Lemma: 4. *(See [44]). Let $p \in P$, $0 < A < 1$; $0 < a < 1$*

$$8A(1-A)[(\alpha\beta - 2\lambda)^2 + (\alpha(A+\alpha) - \beta)^2] + \alpha(1-\alpha)(\beta - 2A\alpha)^2 \le 4\alpha A^2(1-\alpha)^2(1-A). \qquad (24^{**})$$

Then

$$\left|\lambda c_1^4 + Ac_2^2 + 2c_1c_3 - \frac{3}{2}\beta c_1^2c_2 - c_4\right| \le 2.$$

The next section, we reveal the key findings of this investigation with the assistance of Definition 1 and some known lemmas defined in previous sections. The first result established in this paper gives sharp bounds on initial coefficients for the function h belonging to the newly defined class c_{ZN}.The Fekete- Szegö problems are found in Theorem 2. Furthermore, we investigate the second and third Hankel determinants in Theorem 5 and Theorem 6. Finally,we investigate the bounds of logarithmic coefficients in Theorem 7.

Theorem: 1. *Let h has of the form (1) be in the class c_{ZN}. Then*

$$|d_2| \le \frac{1}{2}, |d_3| \le \frac{1}{6}, |d_4| \le \frac{1}{12}, |d_5| \le \frac{1}{20}.$$

Proof: Let $h \in c_{ZN}$, then, by the definition of c_{ZN} ; we have

$$1 + \frac{zh''(z)}{h'(z)} \prec \varphi(z)$$

Utilizing the idea of the Schwarz function, we get

$$1 + \frac{zh''(z)}{h'(z)} \prec \varphi(z) = \frac{1+w(z)}{cosw(z)}. \qquad (26)$$

We define the function $p(z)$ by

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + c_4z^4 + \cdots$$

Or, equivalently

$$w(z) = \frac{p(z)-1}{p(z)+1}$$

$$= \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{4}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{2} + \frac{c_1^3}{8}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1c_3}{2} + \frac{3c_1^2c_2}{8} - \frac{c_2^2}{4} - \frac{c_1^4}{16}\right)z^4 + \cdots \qquad (27^*).$$

By using (27*) together with $\frac{1+w(z)}{cosw(z)}$,we obtain

$$\varphi(w(z)) = 1 + \frac{c_1}{2}z + \left(\frac{c_2}{2} - \frac{c_1^2}{8}\right)z^2 + \left(\frac{c_3}{2} - \frac{c_1c_2}{4} + \frac{c_1^3}{16}\right)z^3 + \left(\frac{c_4}{2} - \frac{c_1c_3}{4} + \frac{c_1^2c_2}{16} + \frac{c_2^2}{8} - \frac{19c_1^4}{1384}\right)z^4 + \cdots \qquad (27)$$

Similarly, we have

$$1 + \frac{zh''(z)}{h'(z)} = 1 + 2d_2z + (6d_3 - 4d_2^2)z^2 + (12d_4 - 18d_2d_3 + 8d_2^3)z^3 + (20d_5 - 18d_3^2 - 32d_2d_4 + 48d_2^2d_3 - 64d_2^4). \qquad (28)$$

Equating the

coefficients of (27) and (28), we obtain

$$d_2 = \frac{c_1}{4}. \qquad (29)$$

$$d_3 = \frac{1}{12}\left(c_2 - \frac{{c_1}^2}{4}\right), \qquad (30)$$

$$d_4 = \frac{1}{24}\left(c_3 + \frac{c_1 c_2}{4} + \frac{{c_1}^3}{16}\right), \qquad (31)$$

And

$$d_5 = \frac{1}{40}\left(-\frac{{c_1}^4}{24} + \frac{{c_2}^2}{2} + \frac{c_1 c_3}{6} - \frac{{c_1}^2 c_2}{48012} + c_4\right). \qquad (32)$$

Now, by applying Lemma 1 to (29) and (30), we obtain

$$|d_2| \leq \frac{1}{2}.$$

And

$$|d_3| \leq \frac{1}{4}.$$

From (31), we have

$$|d_4| = \left|\frac{1}{24}\left[c_3 - 2\left(\frac{1}{8}\right)c_1 c_2 + \frac{{c_1}^3}{16}\right]\right|$$

$$= \left|\frac{1}{24}\left[c_3 - 2Tc_1 c_2 + D{c_1}^3\right]\right|,$$

Where

$$T = \frac{1}{8} \ and \ D = \frac{1}{16}.$$

Hence, we have

$$0 < T < 1 \ and \ T(2T-1) < D < T.$$

Using Lemma 3, we obtain

$$|d_4| \leq \frac{1}{12}.$$

Now, from (32); we have

$$|d_5| = \frac{1}{40}\left|-\frac{{c_1}^4}{24} + \frac{{c_2}^2}{2} + \frac{c_1 c_3}{6} - \frac{{c_1}^2 c_2}{12} + c_4\right|$$

$$= \frac{1}{40}\left|-\frac{{c_1}^4}{24} + \frac{{c_2}^2}{2} + \frac{2c_1 c_3}{12} - \frac{3{c_1}^2 c_2}{2(18)} + c_4\right|$$

$$\frac{1}{40}\left|\lambda {c_1}^4 + A{c_2}^2 + 2\alpha c_1 c_3 - \frac{3\beta {c_1}^2 c_2}{2} + c_4\right|,$$

Where

$$\lambda = \frac{-1}{24}, A = \frac{1}{2}, \alpha = \frac{1}{12}, \beta = \frac{1}{18}.$$

Now, using the left-hand side of (24^{**}), we obtain

$$8A(1-A)[(\alpha\beta - 2\lambda)^2 + (\alpha(A+\alpha) - \beta)^2] + \alpha(1-\alpha)(\beta - 2A\alpha)^2 = \frac{1453}{186624}.$$

And the right-hand side of (24^{**}) is given by

$$4\alpha A^2(1-\alpha)^2(1-A) = \frac{2057}{186624}.$$

We see that the inequality (24^{**}) is satisfied. Therefore, by using Lemmas 5, we obtain

$$|d_5| \leq \frac{1}{20}.$$

Theorem: 2.*Let* $h \in c_{ZN}$.*Then*

$$\left|d_3 - \gamma {d_2}^2\right| \leq \frac{1}{6} max\left\{1, \left|\frac{3\gamma - 1}{2}\right|\right\}$$

Proof: Using (29) and (30), we have

$$\left|d_3 - \gamma {d_2}^2\right| = \frac{1}{12}\left|c_2 - \left(\frac{3\gamma + 1}{4}\right){c_1}^2\right|$$

$$\frac{1}{12}\left|c_2 - v{c_1}^2\right|,$$

Where

$$v = \frac{3\gamma + 1}{4}.$$

Now, applying Lemma 2, we have

$$\left|d_3-\gamma {d_2}^2\right|\le \frac{1}{6}max\left\{1,\left|\frac{3\gamma+1}{2}-1\right|\right\}$$

Upon simplification, the consequent result is as follows:

$$\left|d_3-\gamma {d_2}^2\right|\le \tfrac{1}{6}max\left\{1,\left|\tfrac{3\gamma-1}{2}\right|\right\}. \qquad (38)$$

For [illegible]1 , Theorem 2 yields the following result:

Theorem: 3. *Let the function h of the form (1) belong to the class c_{ZN}, then*

$$\left|d_3-{d_2}^2\right|\le \tfrac{1}{6}. \qquad (39)$$

Theorem: 4. *Let the function h of the form (1), be in the class c_{ZN}, then*

$$|d_2d_3-d_4|\le \frac{1}{12}.$$

The function given by (35), provides a sharp result.

Proof: By using (29),(30) and (31), we obtain

$$|d_2d_3-d_4|=\frac{1}{24}\left|c_3-\frac{1}{4}c_1c_2-\frac{1}{16}{c_1}^3\right|$$

$$=\tfrac{1}{24}\left|c_3-\tfrac{2}{8}c_1c_2-\tfrac{1}{16}{c_1}^3\right|$$

$$=\tfrac{1}{24}\left|c_3-2Tc_1c_2-D{c_1}^3\right|$$

Where

$$T=\tfrac{1}{8}\ and\ D=\tfrac{1}{16}.$$

Hence, we have

$$0<T<1\ and\ T(2T-1)<D<T.$$

By using Lemma 3, we arrive at the required result:

$$|d_2d_3-d_4|\le \tfrac{1}{12}. \qquad (40)$$

Theorem: 5. *Let the function h of the form (1), be in the class c_{ZN}, then*

$$\left|d_2d_4-{d_3}^2\right|\le \frac{1}{36}.$$

The function given by (34) provides a sharp result.

Proof: Using (29),(30) and (31), we obtain

$$\left|d_2d_4-{d_3}^2\right|=\frac{1}{4608}\left({c_1}^4-28{c_1}^2-32{c_2}^2+48c_1c_3\right)$$

After using Lemma 4, we obtain

$$\left|d_2d_4-{d_3}^2\right|=\tfrac{1}{4608}\left|\begin{matrix}-9{c_1}^4-6\left(4-{c_1}^2\right){c_1}^2x-\\ \left(4-{c_1}^2\right)\left[12{c_1}^2+8\left(\left(4-{c_1}^2\right)\right)\right]x^2-24c_1\left(1-|x^2|\right)z\end{matrix}\right| \qquad (41)$$

Let $\eta\in[0,1]$,$|c_1|=c$, $c\in[0,2]$, $|z|=1\ and\ |x|=\eta$, then by taking the moduli and using the triangle inequality, we obtain

$$\left|d_2d_4-{d_3}^2\right|=\frac{1}{4608}\left|\begin{matrix}9c^4+6\left(4-c^2\right)c^2\eta+\\ \left(4-c^2\right)\left[12c^2+8\left(\left(4-c^2\right)\right)\right]\eta^2+24c_1\left(1-\eta^2\right)\end{matrix}\right|$$

$$=\xi(c,\eta)$$

Now, upon setting

$$\frac{\partial\xi(c,\eta)}{\partial\eta}=\frac{1}{4608}\left(6\left(4-c^2\right)c^2+\left(4-c^2\right)\left[24c^2+16\left(\left(4-c^2\right)\right)\right]-48c\right)$$

We observe that $\xi(c,\eta)$ increases on [0,1] with respect to η Therefore, î(c,η) has a maximum value at $\eta=1$ that is,

$$max\xi(c,\eta)=\tfrac{1}{4608}\left(9c^4+6\left(4-c^2\right)c^2+\left(4-c^2\right)\left[12c^2+8\left(\left(4-c^2\right)\right)\right]\right)=B(c) \qquad (42)$$

By differentiating both sides with respect to c, we have

$$B(c)=\tfrac{1}{4608}\left(-4c^3+16c\right).$$

If $B(c)=0$then $c=0$ and

$$c^2=4.$$

For the maximum value, by putting $c = 0$ in (42), we obtain

$$B(0) = \frac{1}{36}$$

Theorem: 6. *Let $h \in c_{ZN}$, then*

$$\left|H_{3,1}(g)\right| \le \frac{43}{2160}.$$

Proof: Since, we have

$$\left|H_{3,1}(g)\right| \le |d_5|\left|d_3 - d_2^2\right| + |d_4||d_4 - d_2 d_3| + |d_3|\left|d_2 d_4 - d_3^2\right|.$$

Now, using Theorem 1, Theorem 3, Theorem 4 and Theorem 5, we obtain

$$\left|H_{3,1}(g)\right| \le \frac{1}{120} + \frac{1}{144} + \frac{1}{216} = \frac{43}{2160}.$$

Hence, the asserted result is proved.

Theorem: 7. *Let the function h of the form (1), belong to the class c_{ZN} ,then*

$$|\beta_1| \le \frac{1}{2}, |\beta_2| \le \frac{1}{12}, |\beta_3| \le \frac{1}{24}, |\beta_4| \le \frac{1}{40}.$$

Proof: Upon substituting values from (29),(30),(31) and (32), we obtain

$$\beta_1 = \frac{c_1}{4} \quad (43)$$

$$\beta_2 = \frac{1}{24}\left(c_2 - \frac{c_1^2}{8}\right), \quad (44)$$

$$\beta_3 = \frac{1}{48}\left(c_3 - \frac{3c_1c_2}{4} + \frac{c_1^3}{16}\right), \quad (45)$$

$$\beta_4 = \frac{-1}{80}\left(\frac{73}{1152}c_1^4 - \frac{13c_2^2}{36} + \frac{1}{4}c_1c_3 - \frac{23c_1^2c_2}{144} - c_4\right). \quad (46)$$

Using the inequality (19) in (43), we obtain

$$|\beta_1| \le \frac{1}{2}.$$

Using the inequality (20) in (44),we obtain

$$|\beta_2| \le \frac{1}{12}.$$

We can rewrite (45) as follows: $|\beta_3| \le \frac{1}{48}\left|c_3 - \frac{3c_1c_2}{4} + \frac{c_1^3}{16}\right|$

$$|\beta_3| = \frac{1}{48}\left|c_3 - \frac{2(3)c_1c_2}{8} + \frac{c_1^3}{16}\right|$$

$$= \frac{1}{48}|c_3 - 2Qc_1c_2 + Rc_1^3|,$$

Where

$$Q = \frac{3}{8} \; and \; R = \frac{1}{16}.$$

Hence, we have

$$0 < Q < 1 \; and \; Q(2Q-1) < R < Q.$$

Now, using Lemma 3, we obtain

$$|\beta_3| \le \frac{1}{24}.$$

Again, we rewrite (46), as follows:

$$|\beta_4| = \frac{1}{80}\left|\frac{73}{1152}c_1^4 - \frac{13c_2^2}{36} + \frac{1}{4}c_1c_3 - \frac{23c_1^2c_2}{144} - c_4\right|$$

$$|\beta_4| = \frac{1}{80}\left|\frac{73}{1152}c_1^4 - \frac{13c_2^2}{36} + 2\left(\frac{1}{8}\right)c_1c_3 + \frac{-3}{2}\left(\frac{23}{216}\right)c_1^2c_2 - c_4\right|$$

$$= \frac{1}{80}\left|\lambda c_1^4 - Ac_2^2 + 2\alpha c_1c_3 + \frac{-3}{2}\beta c_1^2c_2 - c_4\right|, \quad (47)$$

Where

$$\lambda = \frac{73}{1152}, A = \frac{-13}{36}, \qquad \alpha = \frac{1}{8}, \beta = \frac{23}{216}.$$

Next, by using the left-hand side of (24^{**}), we obtain

$$8A(1-A)[(\alpha\beta - 2\lambda)^2 + (\alpha(A+\alpha) - \beta)^2] + \alpha(1-\alpha)(\beta - 2A\alpha)^2 = \frac{115202059}{967458816}$$

And the right-hand side of (24^{**}), is given by

$$4\alpha A^2(1-\alpha)^2(1-A) = \frac{31213}{1327104}.$$

We thus see that the inequality (24^{**}), is satisfied. Therefore, by using Lemma 5, we obtain

$$|\beta_4| \le \frac{1}{40}.$$

Hence, the asserted result is proved.

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