**Physical Education, Health and Social Sciences**
https://journal-of-social-education.org
E-ISSN: 2958-5996
P-ISSN: 2958-5988

**Coefficient Bounds for a Subclass of Bi-Univalent Functions Involving the Salagean Differential Operator and Generalized Telephone Numbers**

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***DOI: https://doi.org/10.63163/jpehss.v3i3.599***

**Abstract:**
This paper investigates a newly defined a specific division of the function set $\Sigma$, That comprises holomorphic and bi-univalent functions in the interior of the unit disk. This subclass is formulated using the Salagean differential operator and incorporates generalized telephone numbers. Moreover, we establish bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$, highlighting the influence of generalized telephone numbers in comparison with bi-univalent function subclasses utilizing the Salagean differential operator in conjunction with the by certain new functions putting in the function class gives some recent findings. Additionally, we discuss the theoretical computational relevance of these function classes in modeling complex systems and their potential applications in algorithmic analysis.

**Keywords**: Bi-univalent functions; Coefficient estimates; Univalent functions; Salagean differential operator; generalized telephone numbers.

## 1. Introduction

The function family A is specified by:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}, \tag{1.1}$$

In the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ the function $f(z)$ is analytic. Consider the subset s of the function class A, consisting of functions that are univalent in $\mathbb{U}$. A fundamental result, as stated in the Koebe one-quarter theorem [4], when f is an element in s, then the image of $\mathbb{U}$ under f necessarily contains a circular region with a radius $\frac{1}{4}$. Consequently, every function f of s possesses a reciprocal function $f^{-1}$, which fulfills:

$$f^{-1}(f(z)) = z, z \in \mathbb{U}. \tag{1.2}$$

For $f(f^{-1}(w)) = w$, for $|w| < r_0(f)$, where $r_0(f) \geq \frac{1}{4}$, The inverse function expansion is given by :

$$f^{-1}(w) = g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{1.3}$$

A function $f \in A$ is termed bi-univalent in $\mathbb{U}$ if both f and its inverse $f^{-1}$ are univalent within $\mathbb{U}$. We define $\Sigma$ as the set of bi-univalent functions in $\mathbb{U}$, following [5, 6].

Various studies have focused on establishing coefficient bounds for subclasses of biunivalent functions [4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17. Despite significant advancements, estimating Taylor-Maclaurin coefficient bounds $|a_n|$ where $n \in \frac{\mathbb{N}}{\{1,2\}}$, and $\mathbb{N} = \{1,2,3,4,5,6 \dots\}$, still lacks a definitive solution.

In 1983, Salagean proposed a differential operator $D^k: A \to A$, defined as:

$$D^0 f(z) = f(z), D^1 f(z) = zf'(z) \tag{1.4}$$

$$D^k f(z) = D\big(D^{k-1} f(z)\big) = z\big(D^{k-1} f(z)\big), k \in \mathbb{N} \tag{1.4}$$

This operator is recursively defined, where applying $D^k$ to a function $f(z)$ involves differentiating $D^{k-1} f(z)$ and multiplying by z. The Salagean differential operator plays a crucial role in the analysis of functions that are analytic, more precisely in quantifying coefficient bounds for specialized groups of functions.

Furthermore, the operator satisfies the relation:

$$D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$$

The primary objective of this work focuses on defining a distinct a specialized group of functions in $\Sigma$ associated with the differential operator formulated by Salagean. Additionally, we deduce coefficient assessments of $|a_2|$ and $|a_3|$ within these newly defined subclasses.

**2. GTN**

The sequence known as involution numbers, often referred to as telephone numbers [2, satisfies the recurrence relation:

$$\mathbb{X}(n) = \mathbb{X}(n-1) + (n-1)\mathbb{X}(n-2), \text{ for } n \geq 2 \tag{2.1}$$

with the initial conditions:

$$\mathbb{X}(0) = \mathbb{X}(1) = 1 \tag{2.2}$$

In the early 19th century, Heinrich August Rothe identified that $Y(n)$ corresponds to the total number of involutions, which are self-inverse permutations, in a symmetric group (see [18, 19]). The connection between involution numbers and symmetric groups was initially recognized in 1800 . Since involutions are closely linked to standard Young tableaux, the number of involutions of order n is also equal to the count of Young tableaux on the set $\{1,2,\dots,n\}$ (see [20] for further details). Later, John Riordan established that the number of possible connection patterns in a telephone system with n subscribers adheres to the given recurrence relation (see [21]).

In 2017, Wloch and Wolowiec-Musial [22] introduced a recurrence relation for integers $n \geq 0$ and $\varphi \geq 1$ to define the generalized telephone numbers (GTN) $\mathbb{X}(\varphi, n)$ :

$$Y(\varphi, n) = \varphi Y(\varphi, n-1) + (n-1)Y(\varphi, n-2), \tag{2.3}$$

with initial conditions:

$$Y(\varphi, 0) = 1, Y(\varphi, 1) = \varphi. \tag{2.4}$$

In 2019, Bednarz and Wolowiec-Musial [23] further extended the concept by proposing an alternative generalization of telephone numbers:

$$Y_\varphi(n) = Y_\varphi(n-1) + \varphi(n-1)Y_\varphi(n-2), \tag{2.5}$$

where the initial conditions are:

$$Y_\varphi(0) = Y_\varphi(1) = 1, \tag{2.6}$$

for $n \geq 2$ and $\varphi \geq 1$. They derived various properties of these numbers, including their generating function, explicit formula, and matrix representations. Additionally, they examined congruence relations and provided new interpretations. More recently, they established the exponential generating function for $Y_\varphi(n)$ along with a summation formula for GTNs:

$$e^{x+\varphi(x^2/2)}=\sum_{n=0}^{\infty} Y_\varphi(n)\frac{x^n}{n!},(\varphi\geq 1) \tag{2.7}$$

Setting $\varphi=1$ yields the classical telephone number sequence $Y(n)$. Some values of $Y_\varphi(n)$ for specific $n$ are as follows:

(1) $Y_\varphi(0)=1$
(2) $Y_\varphi(2)=\varphi+1$
(3) $Y_\varphi(3)=3\varphi+1$
(4) $Y_\varphi(4)=6\varphi+3\varphi^2+1$
(5) $Y_\varphi(5)=10\varphi+15\varphi^2+1$
(6) $Y_\varphi(6)=15\varphi+45\varphi^2+15\varphi^3+1$

Next, consider the function:

$$\Xi(\varphi,\zeta)=e^{(\zeta+\varphi(\zeta^2)/2)}=1+\zeta+\frac{1+\varphi}{2}\zeta^2+\frac{1+3\varphi}{6}\zeta^3+\frac{3\varphi^2+6\varphi+1}{24}\zeta^4+\cdots \tag{2.8}$$

The study of analytic functions defined within the open unit disk $\mathbb{U}$ has been a central focus in mathematical research [24, 25]. Inspired by the work of Srivastava, Mishra, and Gochhayat on bi-univalent functions [32], as well as recent developments involving biunivalent functions related to Gegenbauer polynomials [26], Horadam polynomials [27, 28], conic regions [29], and telephone numbers [24, 25, 30, 31], we introduce a novel subclass of $\Sigma$ associated with GTNs.

In this study, we establish bounds for the coefficients $|a_2|$ and $|a_3|$ for this function class. Additionally, we present new corollaries that extend existing results on bi-univalent functions associated with GTNs, filling a gap in the literature. Our findings contribute to the refinement of coefficient estimates for various subclasses of bi-univalent functions that have been widely studied in recent years.

## 3 .The subclass $S_\Sigma^{s,b}(k,\lambda)$

In this section, we define and analyze a generalized subclass of bi-univalent functions, denoted as $S_\Sigma^{s,b}(k,\lambda)$.

Definition 3.1: Consider two functions $s,b:\mathbb{U}\to\mathbb{C}$ that satisfy the constraint:

$$\min\{\Re(s(\varphi,z)),\Re(b(\varphi,w))\}\geq 0, z\in\mathbb{U}, s(\varphi,0)=b(\varphi,0)=1.$$

Additionally, let f, as defined in (1.1), fall within the analytic function category $\Sigma$. The function f can be described an element of the subclass as

$$f\in S_\Sigma^{s,b}(k,\lambda), k>1, 0\leq\lambda<1$$

if it meets the following criteria:

$$f\in\Sigma \text{ and } \frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z)+\lambda D^k f(z)}\in s(\varphi,z), z\in\mathbb{U} \tag{3.1}$$

And

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w)+\lambda D^k g(w)}\in b(\varphi,w), w\in\mathbb{U}. \tag{3.2}$$

where the function $g(w)$ is given by (1.3).

## 4. Coefficient estimates

For proof of the theorem, we need the following lemma.

**4.1 Lemma:**
(see [4]). If $b \in \mathcal{B}$, consequently, $|c_k| \leq 2$ holds for every k, and $\mathcal{B}$ denotes the collection of functions $b(z)$, which are analytic in $\mathbb{U}$ and fulfill the specified conditions.

$$\Re(b(z)) > 0, b(z) = 1 + c_1 z + c_2 z^2 + \cdots, z \in \mathbb{U}$$

**4.2 Theorem**
Consider f(z) expressed through the series expansion Taylor-Maclaurin (1.1) and classified under $S_{\Sigma}^{s,b}(k,\lambda)$ for $0 \leq \lambda < 1$. Then,

$$|a_2| \leq \min\left\{\frac{1}{2^k(1-\lambda)}, \frac{1+\varphi}{\sqrt{2^{2k+1}(\lambda^2-1)+4\cdot 3^k(1-\lambda)}}\right\} \tag{4.2.1}$$

And

$$|a_3| \leq \min\left\{\frac{2}{2^{2k+1}(1-\lambda^2)} + \frac{1+\varphi}{2\cdot 3^k(1-\lambda)}, \frac{1+\varphi}{2\cdot 3^k(1-\lambda)} + \frac{1+\varphi}{4\cdot 3^k(1-\lambda)+2^{k+1}(\lambda^2-1)}\right\} \tag{4.2.2}$$

Proof: From the conditions (3.1) and (3.2) it follows

$$\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^k f(z)} = s(\varphi, z) \tag{4.2.3}$$

And

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^k g(w)} = b(\varphi, w) \tag{4.2.4}$$

where the function g(w) is defined as in (1.3). The functions $s(\varphi, z)$ and $b(\varphi, w)$ satisfy the conditions outlined in Definition 3.1. Moreover, these functions can be expressed through the following Taylor-Maclaurin series expansions:

$$s(\varphi, z) = 1 + z + \frac{1+\varphi}{2}z^2 + \frac{1+3\varphi}{6}z^3 + \frac{3\varphi^2+6\varphi+1}{24}z^4 + \cdots, z \in \mathbb{U},$$

And

$$b(\varphi, w) = 1 + w + \frac{1+\varphi}{2}w^2 + \frac{1+3\varphi}{6}w^3 + \frac{3\varphi^2+6\varphi+1}{24}w^4 + \cdots, w \in \mathbb{U},$$

By matching the corresponding coefficients in (4.2.3) and (4.2.4), we obtain,

$$2^k(1-\lambda)a_2 = 1 \tag{4.2.5}$$

$$2^{2k}(2-\lambda)a_2^2 + 2\cdot 3^k(1-\lambda)a_3 = \frac{1+\varphi}{2} \tag{4.2.6}$$

$$-2^k(1-\lambda)a_2 = 1 \tag{4.2.7}$$

$$2\cdot 3^k(1-\lambda)(2a_2^2 - a_3) + 2^{2k}(\lambda^2-1)a_2^2 = 2 \tag{4.2.8}$$

From (4.2.5) and (4.2.7), we get

$$2^{2k+1}(1-\lambda)^2 a_2^2 = 2 \tag{4.2.9}$$

Also, from (4.2.6) and (4.2.8), we find that

$$2\left(2^{2k}(\lambda^2-1) + 2\cdot 3^k(1-\lambda)\right)a_2^2 = 1+\varphi \tag{4.2.10}$$

Therefore, we find from the equations (4.2.9) and (4.2.10) that

$$a_2 \leq \frac{1}{2^k(1-\lambda)}$$

and

$$a_2 \leq \frac{1+\varphi}{\sqrt{2^{2k+1}(\lambda^2-1)+4\cdot 3^k(1-\lambda)}}$$

As a result, we obtain the required estimate for the coefficient $a_2$ as presented in (4.2.1).

Furthermore, to determine the bound for the coefficient $a_3$, we subtract equation (4.2.8) from equation (4.6), leading to the following expression:

$$a_3 \leq \frac{2}{2^{2k+1}(1-\lambda^2)} + \frac{1+\varphi}{2\cdot 3^k(1-\lambda)} \tag{4.2.11}$$

Alternatively, by substituting the value of $a_2^2$ from equation (4.10) into the expression derived from subtracting (4.8) from (4.6), we obtain:

$$a_3 \leq \frac{1+\varphi}{2\cdot 3^k(1-\lambda)} + \frac{1+\varphi}{4\cdot 3^k(1-\lambda)+2^{k+1}(\lambda^2-1)} \tag{4.2.12}$$

Thus, the proof of Theorem (4.2) is now fully established.

**5. Definition**

Consider the functions $s, b: \mathbb{U} \to \mathbb{C}$ satisfying the condition:

$$\min\{\Re(s(z)), \Re(b(w))\} \geq 0, z \in \mathbb{U}, s(0) = b(0) = 1$$

Additionally, let f be defined by (1.1) and $g(w)$ by (1.3), where both functions belong to the analytic function class $\Sigma$. We define the subclass:

$$f \in S_{\Sigma}^{s,b}(k,\lambda), k > 1, 0 \leq \lambda < 1$$

provided the following conditions hold:

$$f \in \Sigma \text{ and } \frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^k f(z)} \in s(z), z \in \mathbb{U} \tag{5.1}$$

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^k g(w)} \in b(w), w \in \mathbb{U} \tag{5.2}$$

**5.1 Theorem**

Suppose f(z), expressed through the expansion (1.1), belong to the class $S_{\Sigma}^{s,b}(k,\lambda)$, where $0 \leq \lambda < 1$. Then,

$$|a_2| \leq \min\left\{\frac{1}{2^k(1-\lambda)}, \frac{1}{\sqrt{2^{2k+1}(\lambda^2-1)+4\cdot 3^k(1-\lambda)}}\right\} \tag{5.1.1}$$

And

$$|a_3| \leq \min\left\{\frac{1}{2^{2k}(1-\lambda)^2} + \frac{1}{4\cdot 3^k(1-\lambda)}, \frac{1}{2^{2k+1}(\lambda^2-1)+4\cdot 3^k(1-\lambda)} + \frac{1}{4\cdot 3^k(1-\lambda)}\right\} \tag{5.1.2}$$

Proof: From the conditions (5.1) and (5.2) it follows

$$\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^k f(z)} = s(z) \tag{5.1.3}$$

And

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^k g(w)} = b(w) \tag{5.1.4}$$

where the function $g(w)$ is defined by (1.3), and the functions $s(z), b(w)$ meet the required criteria specified in (5.1). That is,

$$b(z) = s(z) = \frac{1+z}{\cos z} = 1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \cdots \tag{5.1.5}$$

Additionally, the functions $s(z)$ and $b(w)$ can be expressed through their respective Taylor-Maclaurin series expansions as follows:

$$s(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \cdots, z \in \mathbb{U}$$

And

$$b(w) = 1 + w + \frac{1}{2}w^2 + \frac{1}{2}w^3 + \cdots, w \in \mathbb{U}$$

Now, by equating the coefficients in (5.1.3) and (5.1.4), we get

$$2^k(1-\lambda)a_2 = 1, \tag{5.1.6}$$

$$2^{2k}(1-\lambda)a_2^2 + 2\cdot 3^k(1-\lambda)a_3 = \frac{1}{2}, \tag{5.1.7}$$

$$-2^k(1-\lambda)a_2 = 1, \tag{5.1.8}$$

$$2\cdot 3^k(1-\lambda)(2a_2^2 - a_3) + 2^{2k}(\lambda^2-1)a_2^2 = \frac{1}{2}, \tag{5.1.9}$$

From (5.1.6) and (5.1.8), we obtain

$$2^{2k+1}(1-\lambda)^2 a_2^2 = 2 \tag{5.1.10}$$

Also, from (5.1.7) and (5.1.9), we get

$$2\left(2^{2k}(\lambda^2-1) + 2\cdot 3^k(1-\lambda)\right)a_2^2 = \frac{1}{2} \tag{5.1.11}$$

Therefore, we find from equation (5.1.10)

$$|a_2|, l \le \frac{1}{2^k(1-\lambda)} \tag{5.1.12}$$

Also from equation (5.1.11)

$$|a_2| \le \frac{1}{\sqrt{2^{2k+1}(\lambda^2-1) + 4\cdot 3^k(1-\lambda)}} \tag{5.1.13}$$

And in the same way, we subtract (5.1.9) from (5.1.7), thus we get

$$|a_3| \le \frac{1}{2^{2k}(1-\lambda)^2} + \frac{1}{4\cdot 3^k(1-\lambda)} \tag{5.1.14}$$

Moreover, by replacing $a_2^2$ from equation (5.1.11) into the expression derived by subtracting (5.1.9) from (5.1.7), we obtain:

$$|a_3| \le \frac{1}{2\cdot 3^k(\lambda^2-1) + 4\cdot 3^k(1-\lambda)} + \frac{1}{4\cdot 3^k(1-\lambda)} \tag{5.1.15}$$

Thus, the proof of Theorem (5.2) is now complete.

**5.2 Theorem**

Let the function f(z), represented by its Taylor-Maclaurin series expansion in equation (1.1), belong to the class under consideration. $S_\Sigma^{s,b}(k,\lambda), (0 \le \lambda < 1)$. Then,

$$|a_2| \le \min\left\{\frac{1}{2^{k+1}(1-\lambda)}, \frac{1}{\sqrt{3\cdot 2^{2k+2}(\lambda^2-1) + 8\cdot 3^{k+1}(1-\lambda)}}\right\} \tag{5.2.1}$$

And

$$|a_3| \le \min\left\{\frac{1}{2(2^{2k+1}(1-\lambda)^2)} + \frac{1}{24\cdot 3^k(1-\lambda)}, \frac{1}{48\cdot 3^k(1-\lambda)} + \frac{1}{12\cdot 2^{2k+1}(\lambda^2-1) + 48\cdot 3^k(1-\lambda)}\right\} \tag{5.2.2}$$

Proof: From the given conditions in (5.1) and (5.2), we can deduce that:

$$\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^k f(z)} = s(z) \tag{5.2.3}$$

And

$$\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^k g(w)} = b(w) \tag{5.2.4}$$

where the function $g(w)$ is defined in (1.3), and the functions $s(z)$ and $b(w)$ satisfy the conditions outlined in Definitions (5.1) and (5.2). That is,

$$b(z) = s(z) = \frac{z}{\ln(1+z)} = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \cdots \tag{5.2.5}$$

Comparing the coefficients:

$$2^k(1-\lambda)a_2 = 1, \tag{5.2.6}$$

$$2^{2k}(\lambda^2-1)a_2^2 + 2\cdot 3^k(1-\lambda)a_3 = \frac{-1}{12}, \tag{5.2.7}$$

$$-2^k(1-\lambda)a_2 = \frac{1}{2}, \tag{5.2.8}$$

$$2\cdot 3^k(1-\lambda)(2a_2^2 - a_3) + 2^{2k}(\lambda^2-1)a_2^2 = \frac{-1}{12}, \tag{5.2.9}$$

From (5.2.6) and (5.2.8),

$$2^{2k+1}(1-\lambda)^2 a_2^2 = \frac{1}{2} \tag{5.2.10}$$

and

$$2\left(2^{2k}(\lambda^2-1) + 2\cdot 3^k(1-\lambda)\right)a_2^2 = \frac{-1}{6}, \tag{5.2.11}$$

Now, solving for $a_2$ using equations (5.2.10) and (5.2.11), we obtain:

$$|a_2| \le \frac{1}{2^{k+1}(1-\lambda)} \tag{5.2.12}$$

and

$$|a_2| \le \frac{1}{\sqrt{3\cdot 2^{2k+2}(\lambda^2-1) + 8\cdot 3^{k+1}(1-\lambda)}} \tag{5.2.13}$$

On the other hand, substituting the value of $a_2^2$ from equation (5.2.11) into the equation derived by subtracting (5.2.9) from (5.2.7), we obtain:

$$|a_3| \le \frac{1}{48\cdot 3^k(1-\lambda)} + \frac{1}{12\cdot 2^{2k+1}(\lambda^2-1) + 48\cdot 3^k(1-\lambda)} \tag{5.2.14}$$

This concludes the proof of Theorem (5.2).

**Conclusion**

This study presents an exploration of a newly defined subset within the class Σ, which comprises holomorphic and bi-univalent functions connected to the Salagean differential operator and generalized telephone numbers. By determining coefficient estimates for $|a_2|$ and $|a_3|$, we compared the properties of generalized telephone numbers alongside a specific subclass of bi-univalent functions. Our results extend and generalize recent findings in the literature, providing deeper insights into the interplay between the Salagean differential operator and analytic bi-univalent function classes. Future research may focus on further refining these coefficient estimates and exploring their implications in geometric function theory. The recursive nature of GTNs and operator-based function transformations hint at potential intersections with symbolic computation and complexity theory in computer science, providing scope for future interdisciplinary exploration.

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