Physical Education, Health and Social Sciences

https://journal-of-social-education.org

E-ISSN: 2958-5996
P-ISSN: 2958-5988

Determining Upper Estimate for The Third Hankel Determinant of Bi Univalent Function Associated with A New Function.

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DOI: https://doi.org/10.63163/jpehss.v3i3.585

Abstract:
This research explores the characteristics of bi univalent functions related with a new function, with focus on establishing upper estimates for the hankel determinants $H_3(1)$.Our analysis provides deeper understandings of the properties of these functions , advancing our understandings of bi univalent function theory and its connections to geometric function theory.

Introduction

Let A be the family of analytic functions defined by the form

$$f(z) = z + \sum_{n=2}^{\infty} r_n z^n \tag{1}$$

In the disk $U = \{z \in C: |z| < 1\}$. We examine the class A of functions that are analytic and its subclasses S of univalent functions. The function f is considered bi univalent within S, if its inverse g also lies in S, and satisfying $g(f(z)) = z$ and $f(g(w)) = w \left(z \in U, |w| < R(f); R(f) > \frac{1}{4}\right)$ and given by

$$g(w) = w - r_2 w^2 + (2r_2^2 - r_3)w^3 - (5r_2^3 - 5r_2 r_3 + r_4)w^4 \dots .... \tag{2}$$

Let f be function in $\boldsymbol{S}$, and its inverse g can be extended as a function belonging to S, then f is called bi univalent in and denoted by Σ. Subclasses of Σ namely bi starlike (or biconvex) of order $0 \leq \theta < 1$ and have been establish by Brannan and Taha cite: 1 theses classes are non sharp coefficient estimates $|r_2|$ and $|r_3|$ [1,2] . However the nth Taylors and Maclurin coefficients $|r_n|(n \in (3,4 \dots))$ and remain an unresolved challenge [1-5].
The pioneering work of Srivastava et al, [6] has significantly revitalized the study of bi univalent functions in recent years, for a concise historical overview and further details refer to [1-5], [7-12] we observe that $\Sigma \neq \phi$ and $\Sigma \subseteq S$, in this context we define the subordination as follow an analytic function f is said to be subordinate to another function g if there exists an analytic function $w: U \to U$ with $w(0) = 0$ fulfilling $f(z) = g(w(z))$ for all $z \in U$. And it is denoted by

$$f \prec g$$

Ma and Minda studied the integration of various subclasses of star like and convex function, they examined the scenario where one of the function specifically $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$, is subordinate to another holomorphic function φ. To achieve this, they analyze a holomorphic function φ define on the unit disk U and mapping it onto the complex plane C satisfying the following conditions

1. $\varphi'(0) > 0$.
2. φ is univalent in the unit disk U.
3. The image of U under φ is star like with respect to 1 .
4) The image of U under ϕ exhibits symmetry about the real axis.

Ravichandran and Kumar [14] introduced the class of $RS^*(\alpha)$ of star like function of reciprocal order $\alpha (0 \leq \alpha \leq 1)$ specified by the requirement,

$$\text{Re}\left(\frac{f(z)}{zf'(z)}\right) > \alpha.$$

This class is generalization of star like functions and its properties have been extensively studied, Ma and Minda works on bi star like and bi convex functions provides a frame work for analyzing functions that are both star like and convex.
In the present work we propose the new subclass of bi univalent functions, represented by $RS_{\Sigma}^*(\lambda)$ (refer to definition 2) which is Ma Minda type starlike function, specifically we consider the function f that satisfies the constraints 1-4 mentioned earlier. the function

$$\varphi(z) = 1 + \frac{z}{1 - z} \tag{3}$$

Definition 1.(see Raina and Sokot [15]; see also [16-18]. Let S* the family of functions f belonging to A which fulfills the subordination condition

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{z}{1-z}$$

Noonan and Thomas [19] investigated the qth Hankel determinant for an holomorphic function $f(z)$ with the expression

$$f(z) = z + \sum_{n=2}^{\infty} r_n z^n.$$

By

$$H_q(n) = \begin{vmatrix} r_n & r_{n+1} & \cdots & r_{n+q-1} \\ r_{n+1} & r_{n+2} & \cdots & r_{n+q} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ r_{n+q-1} & r_{n+q} & \cdots & r_{n+2q-2} \end{vmatrix}, (q \geq 1)$$

In particular we have

$$H_3(1) = \begin{vmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_4 \\ r_3 & r_4 & r_5 \end{vmatrix}, (r = 1),$$

by applying the triangular in equality for $H_3(1)$, we obtain

$$|H_3(1)| \leq |r_3||r_2 r_4 - r_3^2| - |r_4||r_4 - r_2 r_3| + |r_5|\,| \qquad (4)$$

Fakete and Szego [20] considered the well-known functional $H_2(1)$ [21, 22].
Their early work focused on estimating of $|r_3 - \beta r_2^2|$, with $\beta \in R; f \in A$, then

$$|r_3 - \beta r_2^2| = \begin{cases} 4\beta - 3, & \beta \geq 1 \\ 1 + 2e^{\left(-\frac{2\beta}{1-\beta}\right.}, & 0 \leq \beta \leq 1 \\ 3 - 4\beta, & \beta \leq 0 \end{cases}$$

Inspired by the investigation of the second Hankel determinant corresponding to distinct subclasses of Σ [23_30], this paper provides estimation for the upper bounds of $H_3(1)$ for the function with in the class $RS_{\Sigma}^{*}(\lambda)$.

Objectives: This research investigate the third Hankel determinant associated with bi univalent function related with a new function and explore the practical application, aiming to uncover the significance and potential impacts of these function in various real world contexts.

2. About: $RS_{\Sigma}^{*}(\lambda)$; In this section we will establish the a sub class of Σ associated with a new function

Definition: Consider 0 ≤ λ ≤ 1. A function f (z) (1).in the class Σ , is define to be in the class $RS_{\Sigma}^{*}(\lambda)$. if it fulfills the following subordinations:

$$\frac{\lambda z f'(z) + (1-\lambda) f(z)}{\lambda z^2 f''(z) + z f'(z)} \prec \varphi(z)$$

(5)

and

$$\frac{\lambda w g'(w) + (1-\lambda) g(w)}{\lambda w^2 g''(w) + w g'(w)} \prec \varphi(w) \qquad (6)$$

where $z, w \in U$ and $g = f^{-1}$

3. Hankel Estimate of $RS_{\Sigma}^{*}(\lambda)$;

The following lemma is required for deriving preliminary bounds and for addressing the Faketo-Szego problem.

Lemma 1; [36]. Consider p as the class of all holomorphic functions p(z) having the structure

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n$$

(7)

With $\mathrm{Re}(p(z)) > 0$ for all $z \in U$.

Then $|p_n| \leq 2$, for $n \in N$

Lemma2:. Suppose $p(z)=1+p_1z+p_2z^2+\ldots \in P$, then

$$2p_2 = p_1^2 + t(4 - p_1^2)$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 t - (4 - p_1^2)p_1 t^2 + 2(4 - p_1^2)(1 - |t|^2)z$$

for some t, y with $|t| \leq 1, |y| \leq 1$.

Theorem 1. Let f(z) be given by(1) be in the class $RS_{\Sigma}^{*}(\lambda)$; ,0≤λ≤1. Then we have

$$|r_2 r_4 - r_3^2| \leq \frac{14 + 42\lambda - \lambda^2 - 2\lambda^3}{64(1+\lambda)^4(1+3\lambda)}. \qquad (8)$$

Proof: Let f be in the class$RS_{\Sigma}^{*}(\lambda)$. $\exists\, u, v; U \to U$ with $u(0) = v(0), |u(z)| < 1, |v(w)| < 1$, where

$$\frac{\lambda z f'(z) + (1-\lambda) f(z)}{\lambda z^2 f''(z) + z f'(z)} \prec \varphi(z). \tag{9}$$

and

$$\frac{\lambda w g'(w) + (1-\lambda) g(w)}{\lambda w^2 g''(w) + w g'(w)} \prec \varphi(w) \quad . \tag{10}$$

Consider the function $p, q \in P$ with expressions

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + \sum_{n=2}^{\infty} p_n z^n$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + \sum_{n=2}^{\infty} q_n w^n$$

it follows that

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 - p_1 p_2 + \frac{p_1^3}{4}\right) z^3 + \cdots. \right. \tag{11}$$

And

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \left(q_3 - q_1 q_2 + \frac{q_1^3}{4}\right) w^3 + \cdots. \right. \tag{12}$$

substituting (11)and (12) in (3), we obtain

$$\varphi\big(u(z)\big) = 1 + \frac{p_1 z}{2} + \left(\frac{p_2}{2} + \frac{p_1^2}{8}\right) z^2 + \left(\frac{p_3}{2} + \frac{p_1^3}{8}\right) z^3 + \left(\frac{p_4}{2} + \frac{p_1^2 p_2}{8} - \frac{p_1^4}{64}\right) z^4 + \cdots. \quad (13).$$

and

$$\varphi\big(v(w)\big) = 1 + \frac{q_1 w}{2} + \left(\frac{q_2}{2} + \frac{q_1^2}{8}\right) w^2 + \left(\frac{q_3}{2} + \frac{q_1^3}{8}\right) w^3 + \left(\frac{q_4}{2} + \frac{q_1^2 q_2}{8} - \frac{q_1^4}{64}\right) w^4 \ldots. \tag{14}$$

since $f \in \sum$ has a maclurin series define by (1) computation show that the inverse $g = f^{-1}$ has the expansion by (2) and we have

$$\begin{aligned}
\frac{\lambda z f'(z) + (1-\lambda) f(z)}{\lambda z^2 f''(z) + z f'(z)} &= 1 - (1+\lambda) r_2 z + 2[(1+\lambda)^2 - (1+2\lambda) r_3] z^2 \\
&+ [7(1+3\lambda+2\lambda^2) r_2 r_3 - 4(1+3\lambda+3\lambda^2+\lambda^4) r_2^3 - 3(1+3\lambda) r_4] z^3 \\
&+ 2[5(1+4\lambda+3\lambda^2) r_2 r_4 - 10(1+4\lambda+5\lambda^2+2\lambda^3) r_3 r_2^2 + \\
&3(1+4\lambda+4\lambda^2) r_3^2 + 4(1+4\lambda+6\lambda^2+4\lambda^3+\lambda^4) r_2^4 - 2(1+4\lambda) r_5] z^4 + \cdots
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\frac{\lambda w g'(w) + (1-\lambda) g(w)}{\lambda w^2 g''(w) + w g'(w)} &= 1 - (1+\lambda) r_2 w + 2[(1+2\lambda) r_3 - (1+2\lambda-\lambda^2) r_2^2] w^2 \\
&+ [(5+15\lambda+10\lambda^2+4\lambda^3) r_2^3 - (8+24\lambda+\lambda^2) r_2 r_3 + 3(1+3\lambda) r_4] w^3 \\
&+ 2[42(1+4\lambda) r_2 r_3 - 3(1+4\lambda-4\lambda^2) r_3^2 - (7+28\lambda-15\lambda^2) r_2 r_4 - \\
&(5+28\lambda-47\lambda^2+24\lambda^3-4\lambda^4) r_2^4 - (27+64\lambda+73\lambda^2-20\lambda^3) r_3 r_2^2 \\
&+ 2(1+4\lambda) r_5] w^4 + \cdots
\end{aligned} \tag{16}$$

From (13)and (15) we obtain

$$-(1+\lambda) r_2 = \frac{p_1}{2}. \tag{17}$$

$$2[(1+\lambda)^2 - (1+2\lambda) r_3] = \frac{p_2}{2} + \frac{p_1^2}{8}. \tag{18}$$

$$[7(1+3\lambda+2\lambda^2) r_2 r_3 - 4(1+3\lambda+3\lambda^2+\lambda^4) r_2^3 - 3(1+3\lambda) r_4] = \frac{p_3}{2} + \frac{p_1^3}{8}. \tag{19}$$

$$\begin{aligned}
&[(5+28\lambda-47\lambda^2+24\lambda^3-4\lambda^4) r_2^4 - (27+64\lambda+73\lambda^2-20\lambda^3) r_3 r_2^2 \\
&+4(1+4\lambda+6\lambda^2+4\lambda^3+\lambda^4) r_2^4 = \left(\frac{p^4}{2}\right) + \left(\frac{p^{12} p^2}{8}\right) - \left(\frac{p^{14}}{64}\right).
\end{aligned} \tag{20}$$

Moreover from (13) and (15), we get

$$-(1+\lambda)r_2 = \frac{q_1}{2} \quad (21)$$

$$2[(1+2\lambda)r_3 - (1+2\lambda-\lambda^2)r_2^2] = \frac{q_2}{2} + \frac{q_1^2}{8} \quad (22)$$

$$[(5+15\lambda+10\lambda^2+4\lambda^3)r_2^3 - (8+24\lambda+\lambda^2)r_2r_3 + 3(1+3\lambda)r_4] = \frac{q_3}{2} + \frac{q_1^3}{8} \quad (23)$$

$$2[42(1+4\lambda)r_2r_3 - 3(1+4\lambda-4\lambda^2)r_3^2 - (7+28\lambda-15\lambda^2)r_2r_4 -$$
$$(5+28\lambda-47\lambda^2+24\lambda^3-4\lambda^4)r_2^4 - (27+64\lambda+73\lambda^2-20\lambda^3)r_3r_2^2 + 2(1+4\lambda)] = \frac{q_4}{2} + \frac{q_1^2q_2}{8} - \frac{q_1^4}{64} \quad (24)$$

It follows from (17) and (21) that

$$r_2 = \frac{-p_1}{2(1+\lambda)} = \frac{q_1}{2(1+\lambda)}. \quad (25)$$

i.e

$$-p_1 = -q_1 \quad (26)$$

Subtracting (23) from (18) and considering (24), we obtain

$$r_3 = \frac{p_1^2}{4(1+\lambda)^2} - \frac{p_2-q_2}{8(1+2\lambda)}. \quad (27)$$

Moreover subtracting (23) from (19) we obtain and considering ref: 24 and (26) we obtain

$$r_4 = \frac{5(1+3\lambda+\lambda^2)p_1(p_2-q_2)}{32(1+\lambda)(1+2\lambda)(1+3\lambda)} - \frac{(-6+18\lambda-7\lambda^2-8\lambda^3)p_1^3}{48(1+\lambda)^3(1+3\lambda)} - \frac{p_3-q_3}{12(1+3\lambda)}$$
$$-\frac{p_1^3-q_1^3}{48(1+3\lambda)} \quad (28)$$

More ever from (25),(27) and (28), It follows that

$$r_2r_4 - r_3^2 = \frac{-5(1+3\lambda+\lambda^2)p_1^2(p_2-q_2)}{64(1+\lambda)^2(1+2\lambda)(1+3\lambda)} + \frac{(-6+18\lambda-7\lambda^2-8\lambda^3)p_1^3}{96(1+\lambda)^4(1+3\lambda)} + \frac{p_1(p_3-q_3)}{24(1+3\lambda)(1+\lambda)}$$
$$+\frac{p_1^4}{48(1+\lambda)(1+3\lambda)} - \frac{p_1^4}{16(1+\lambda)^4} + \frac{p_1^2(p_2-q_2)}{16(1+\lambda)^2(1+2\lambda)} - -\frac{(p_2-q_2)^2}{64(1+2\lambda)^2} \quad (29)$$

Now According to (ref: lema1)and (lemma 2) and equation (26) we obtain

$$p_2 - q_2 = \frac{4-p_1^2-}{2}(t-y), p_2+q_2 = p_1^2 + \frac{4-p_1^2-}{2}(t+y)$$
$$p_3 - q_3 = \frac{p_1^3}{3} + \frac{(4-p_1^2)p_1-}{2}(t+y) - \frac{(4-p_1^2)p_1-}{4}(t^2+y^2) +$$
$$\frac{(4-p_1^2)-}{2}(1-|t|^2)z - (1-|y|^2)w \quad (30)$$

For particular x, y, z and w with $|t| \le 1, |y| \le 1, |z| \le 1$ and $|w| \le 1$

Given that $p \in P$, it follows that $|p_1| < 2$, l by setting $p_1 = p$, we can assume with out affecting of generality that $p \in [0,2]$, thus substituting equation (ref: 29) in (ref: 28) we obtain

$$|r_2r_4 - r_3^2| \le H_1 + H_2(\gamma+\delta) + H_3(\gamma^2+\delta^2) + H_4(\gamma+\delta)^2 = H(\gamma,\delta)$$

where

$$H_1 = H_1(\lambda,p) = \frac{(14+42\lambda-\lambda^2-2\lambda^3)p^4}{(1+\lambda)^4(1+3\lambda)} \ge 0$$

$$H_2 = H_2(\lambda,p) = \frac{(47+141\lambda+31\lambda^2)p^2(4-p^2)}{384(1+\lambda)^2(1+2\lambda)(1+3\lambda)} \ge 0$$

$$H_3 = H_3(\lambda,p) = \frac{(p-2)p(4-p^2)}{96(1+3\lambda)(1+\lambda)} \le 0$$

$$H_4 = H_4(\lambda,p) = \frac{(4-p^2)^2}{256(1+2\lambda)^2} \ge 0$$

our goal is to maximize $H(\gamma,\delta)$ over the domain $[0,1] * [0,1]$ subjected to $p \in [0,2]$, $H_3 + 2H_2 \ge 0$ and $H_3 \le 0$, and our analysis yields $p \in [0,2]$ because

$$H_{\gamma\gamma}H_{\delta\delta} - \left(H_{\gamma\delta}\right)^2 < 0$$

Thus, the function H cannot have a local maximum in the interior of the closed square. Now, we investigate the maximum of H on the boundary oH the closed square, such that $\gamma = 0$ and $0 \le \delta \le 1$, and we obtain

$$H(0,\delta) = \Phi(\delta) = H_1 + H_2\delta + (H_3+H_4)\delta^2$$

we now consider two cases.

Case1

$$\Phi'(\delta) = H_2 + 2(H_3+H_4) > 0$$

i.e., $\Phi(\delta)$ is an nondecreasing function Hence, for fixed $p \in [0,2$ the peak value of $\Phi(\delta)$ occurs
at $\delta = 1$ and

$$\max\Phi(\delta) = \Phi(1) = H_1 + H_2 + H_3 + H_4$$

Case2
$H_3 + H_4 < 0$, because $2(H3 + H4) + H2 \geq 0, 0 < \delta < 1$, where $0 < p < 2$, and it is evident that

$$2(H_3 + H_4) + H_2 < 2(H_3 + H_4)\delta + H_2 < H_2$$

Since $\Phi(\delta) > 0$. Thus, the maximum of $\Phi(\delta)$ occurs at $\gamma = 1$ and $0 \leq \delta \leq 1$, and we obtain

$$H(1,\delta) = \theta(\delta) = (H_3 + H_4)\delta + (H_2 + 2H_4)\delta + H_1 + H_2 + H_3 + H_4$$

so, from the cases of $H_3 + H_4$, we obtain

$$\max\theta(\delta) = \theta(1) = H_1 + 2H_2 + 2H_3 + 4H_4$$

Since $\Phi(1) \leq \theta(1)$, we gain $\max(H(\gamma,\delta)) = H(1,1)$ on the perimeter of the $[0,1] \times [0,1]$. let T be a real valued function over (0,1) by

$$T(p) = \max(H(\gamma,\delta)) = H(1,1) = H_1 + 2H_2 + 2H_3 + 4H_4.$$

Placing H_1, H_2, H_3 and H_4 in the function T, we obtain

$$T(p) = S + R + Q$$

where

$$S = \frac{(14 + 42\lambda - \lambda^2 - 2\lambda^3)p^4}{(1+\lambda)^4(1+3\lambda)}$$

$$R = \frac{(p-2)p(4-p^2)}{48(1+3\lambda)(1+\lambda)}$$

and

$$Q = \frac{[(47 + 141\lambda + 31\lambda^2)(1+2\lambda) + 3(1+\lambda)^2(1+3\lambda)]p^2(4-p^2)}{192(1+\lambda)^2(1+2\lambda)(1+3\lambda)}$$

Our calculation showed that T(p) is an increasing function, yielding the maximum at p=2

$$\max T(p) = T(2) = \frac{14+42\lambda-\lambda^2-2\lambda^3}{64(1+\lambda)^4(1+3\lambda)} \quad (31)$$

Consequently, the proof is finish.

Theorem 2. Let $f(z)$ be given by ref: 1 belongs to $RS^*_\Sigma(\lambda)$,$0 \leq \lambda \leq 1$. Then we have

$$|r_2r_3 - r_4| \leq \begin{cases} \frac{4 + 12\lambda + 5\lambda^2 - 4\lambda^3}{6(1+3\lambda)(1+\lambda)^3}, & m \leq p < 2 \\ \frac{1}{6(1+3\lambda)}, & 0 \leq p \leq m \end{cases} \quad (32)$$

where

$$m = \frac{-s_3 \pm \sqrt{s_3^2 - 12(s_1 - s_2)s_2}}{3(m_1 - m_2)}$$

$$s_1 = \frac{4 + 12\lambda + 5\lambda^2 - 4\lambda^3}{48(1+3\lambda)(1+\lambda)^3}$$

$$s_2 = \frac{(-1 + 87\lambda + \lambda^2) + 4(1+\lambda)(1+2\lambda)}{96(1+\lambda)(1+2\lambda)(1+3\lambda)}$$

$$s_3 = \frac{1}{12(1+3\lambda)}$$

Proof : from (25), (27) and (28), we obtain

$$|r_2r_3 - r_4| \leq \left|\frac{(2 + 6\lambda - \lambda^2 - 6\lambda^3)p_1^3}{48(1+3\lambda)(1+\lambda)^3} + \frac{(-3 + 21\lambda - 5\lambda^2)p_1(p_2 - q_2)}{32(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{p_3 - q_3}{12(1+3\lambda)}\right|$$

Now inlight of lemma 2, we can assume without restriction, that that $p \in [0,2]$, such that $p_1 = p$. Therefore, for ζ_1=|t| and ζ_2=|y|, we get

$$|r_2r_3 - r_4| \leq F_1 + F_2(\zeta_1 + \zeta_2) + F_3(\zeta_1 + \zeta_2) = F(\zeta_1,\zeta_2)$$

where

$$F_1(\lambda,p) = \frac{(8 - 3\lambda + 2\lambda^2)p^3}{48(1+\lambda)^3(1+3\lambda)} \geq 0$$

$$F_2(\lambda,p) = \frac{(7 + 66\lambda + 32\lambda^2)(4-p^2)p}{192(1+\lambda)(1+3\lambda)(1+2\lambda)} \geq 0$$

$$F_3(\lambda,p) = \frac{(4-p^2)(p-2)}{48(1+3\lambda)} \leq 0$$

We use the same proof technique as in theorem 1 . Thus, the maximum occurs at $\zeta_1 = 1$ and $\zeta_2 = 1$ in closed square [0,2],

$$\theta(p) = \max\left(F(\zeta_1, \zeta_2)\right) = F_1 + 2(F_2 + F_3).$$

Putting F_1, F_2 and F_3 in θ(p), we obtain

$$\theta(p) = s_1 p^3 + s_2 p(4 - p^2) + s_3(4 - p^2)$$

where

$$s_1 = \frac{8 - 3\lambda + 2\lambda^2}{48(1+\lambda)^3(1+3\lambda)}$$

$$s_2 = \frac{7 + 66\lambda + 32\lambda^2) + 4(1+\lambda)(1+2\lambda))}{192(1+\lambda)(1+3\lambda)(1+2\lambda)}$$

and

$$s_3 = \frac{1}{12(1+3\lambda)}$$

Therefore,

$$\theta'(p) = 3(s_1 - s_2)p^2 + 2s_3 p + 4s_2$$
$$\theta''(p) = 6(s_1 - s_2)p + 2s_3$$

Suppose $s_1 - s_2 > 0$, it follows that $s_1 > s_2$. which implies, $\theta'(p)' > 0$, ,as a result θ(p) is an ascending function on [0,2]. and Therefore, achives its maximum at p=2, i.e.,

$$|r_2 r_3 - r_4| \le \theta(2) = \frac{4 + 12\lambda + 5\lambda^2 - 4\lambda^3}{6(1+\lambda)^3(1+3\lambda)}$$

conversely, if $s_1 - s_2 < 0$ with $\theta'(p) = 0$, the following results are obtained

$$p = m = \frac{-s_3 \pm \sqrt{s_3^2 - 12(s_1 - s_2)s_2}}{3(s_1 - s_2)}$$

for m<p ≤ 2. and θ′(p) > 0, implies that θ(p) increases resulting in the maximum value of θ(p) on [0,2]. being attained at p=2, implies that θ(p) is falling on [0,2], and thus, θ(p) achieves its highest value at p=0,
i.e.,

$$|r_2 r_3 - r_4| \le \theta(0) = \frac{1}{6(1+3\lambda)}$$

This complete the proof.

Theorem 3 . Let $f(z)$) (ref: 1) $\in RS_\Sigma^*(\lambda)$,$0 \le \lambda \le 1$. Then we have

$$|r_3 - r_2^2| \le \frac{1}{2(1+2\lambda)} \tag{33}$$

$$|r_3| \le \frac{1}{(1+\lambda)^2} + \frac{1}{2(1+2\lambda)} \tag{34}$$

Proof. By using (27) and ref: lema1 we obtain (34)
we examine the underlying Fekete--Szegö functional, for $\mu \in C$ and $f(z) \in RS_\Sigma^*(\lambda)$

$$|r_3 - \mu r_2^2| = \frac{p_1^2}{4(1+\lambda)^2}(1-\mu) + \frac{p_2 - q_2}{8(1+2\lambda)}$$

By Lemma 1, we obtain

$$|r_3 - \mu r_2^2| \le \frac{(1-\mu)}{(1+\lambda)^2} + \frac{1}{2(1+2\lambda)}$$

for $\mu = 1$, we gain (33)

Theorem. Let $f(z) \in RS_\Sigma^*(\lambda)$,$0 \le \lambda \le 1$. Then, we have

$$|r_4| \le \frac{5(1+3\lambda+\lambda^2)}{4(1+\lambda)(1+2\lambda)(1+3\lambda)} + \frac{(-6+18\lambda-7\lambda^2-8\lambda^3)}{6(1+\lambda)^3(1+3\lambda)} + \frac{2}{3(1+3\lambda)} \tag{35}$$

$$|r_5| \le \frac{(42+300\lambda+416\lambda^2-162\lambda^3+72\lambda^4)}{4(1+\lambda)^4(1+3\lambda)(1+4\lambda)} + \frac{(66+390\lambda-426\lambda^2+126\lambda^3-24\lambda^4)}{16(1+\lambda)^2(1+2\lambda)(1+3\lambda)(1+4\lambda)} + \frac{21}{16(1+\lambda)^2} + \frac{3}{(1+2\lambda)^2} + \frac{1}{4(1+4\lambda)} + \frac{21}{4(1+2\lambda)(1+\lambda)} \tag{36}$$

Proof. From (30) and by Lemma 2, we obtain (36). Subtracting (26) from (22), we have

$$8(1+4\lambda)r_5 = (24+96\lambda)r_2 r_4 + (24+48\lambda+46\lambda^2)r_3 r_2^2 + (12+48\lambda)r_3^2 + (18+88\lambda-54\lambda^2+80\lambda^3)r_2^4 - 84(1+4\lambda)r_2 r_3 - \frac{p_4 - q_4}{2} + \frac{p_1^2(p_2 - q_2)}{8} \tag{37}$$

.Substituting properly (25), (27) and (31), we have

$$8(1+4\lambda)r_5=\frac{(42+300\lambda+416\lambda^2-162\lambda^3+72\lambda^4)p_1^4}{8(1+\lambda)^4(1+3\lambda)}+\frac{84(1+4\lambda)p_1^3}{8(1+\lambda)^3}-\frac{(66+390\lambda-426\lambda^2+126\lambda^3-24\lambda^4)p_1^2(p_2-q_2)}{32(1+\lambda)^2(1+2\lambda)(1+3\lambda)}+\frac{(p_2-q_2)^2(12+48\lambda)}{16(1+2\lambda)^2}+\frac{84(1+4\lambda)p_1(p_2-q_2)}{16(1+\lambda)(1+2\lambda)}+\frac{p_4-q_4}{2}$$

By applying ref: lema1, we obtain (36)

Theorem 5. Let $f(z)\in RS_{\Sigma}^{*}(\lambda)$,$0\leq\lambda\leq 1$. Then, we have

$$H_3(1)\leq\left\{\begin{matrix} KK_1-K_2\left(\frac{4+12\lambda+5\lambda^2-4\lambda^3}{6(1+\lambda)^3(1+3\lambda)}\right)+K_3K_4 & m\leq p\leq 2 \\ KK_1-K_2\left(\frac{1}{6(1+3\lambda)}\right)+K_3K_4 & 0<p\leq m \end{matrix}\right\} \quad (38)$$

where $K, K_1, K_2, K_3, K_4, and\ m$ are given by (34) , (8) , (35) , (36) , and (33) , respectively.

Proof Since

$$|H_3(1)|\leq|r_3||r_2r_4-r_3^2|-|r_4||r_4-r_2r_3|+|r_5|\,|$$

Substituting (8), (33), (36) and (37) in (4) we obtain (38)

Conclusion

These functions are closely related to the Koebe function, a fundamental extremal function in geometric function theory , by investigating the third Hankel determinants of bi univalent functions, we gain valuable insights into their properties and behaviour.The finding of this research provide new prospective on the associated determinants of bi univalent functions, shedding light on their characteristics and constraints. This study's results have important implications for the study of complex analysis, particularly in the context of bi univalent functions and their connections to the koebe function .This research contributes significantly to the field of complex analysis by advancing our understanding of bi univalent functions.

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