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# On the Domination Number of the Splitting Graph of $C_{\xi}^{\kappa}$

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**Abstract**
This paper explores the domination number of the splitting graph associated with higher powers of cycle graphs. We begin by computing the domination number of the splitting graph of the square and cube of the cycle graph $C_{\xi}$. Through analysis and observed patterns, we propose a general conjecture regarding the domination number of the splitting graph of the $\kappa^{th}$ power of $C_{\xi}$. These results extend the existing work on domination parameters in transformed graphs and provide a foundation for further theoretical development and applications.

## 1. Introduction

Graph theory has experienced rapid development in recent decades, finding applications across a range of disciplines including computer science, biology, linguistics, and social sciences. Among its many branches, the study of domination in graphs stands out as a central area due to both its theoretical appeal and practical significance.

The concept of domination in graphs dates back to the 19th century, when de Jaenisch (1862) investigated the problem of placing the minimum number of queens to dominate all squares on an n×n chessboard [1]. A formal framework for domination emerged in the mid-20th century. Claude Berge (1958) introduced the idea of the "coefficient of external stability", which we now recognize as the domination number of a graph [2]. Ore (1962) refined this concept by formally defining the domination number $\gamma(G)$ and the notion of a dominating set [3].

Further significant contributions were made by Cockayne and Hedetniemi (1977), who introduced the notation $\gamma(G)$ for the domination number and compiled foundational results that sparked an explosion of research in the field [7]. In the decades that followed, extensive studies have focused on domination in various classes of graphs, including trees, cycles, and their generalizations.

Important progress has also been made in identifying bounds for the domination number. Shepard et al. (1989) [10] and Reed (1996) [11] refined bounds for connected graphs. Behzad et al. (2007) [12] examined Petersen graphs $P(\xi, \kappa)$, while Vaidya et al. (2012) [13] explored domination in splitting graphs of paths and cycles. Kazemnejad et al. (2019, 2020) [14, 15] studied total and middle domination in transformed graphs. More recently, Murugan et al. (2022) [16] provided bounds for domination numbers of square graphs. Domination theory has also been discussed from broader perspectives in works like those by Guichard [17], Brigham et al. [18], and Ribeiro et al. [19], highlighting its interdisciplinary applications and algorithmic challenges.

Our current work continues in this direction by considering the splitting graph of the $k^{th}$ power of cycle graphs. The $\kappa^{th}$ power $G^{\kappa}$ of a graph G connects any two vertices with distance at most κ, while the splitting graph Sp(G) is formed by inserting a new vertex for every edge in G and connecting it to the end points of that edge.
In this paper, we calculate the domination number $\gamma(Sp(C_{\xi}^{\kappa}))$ for $\kappa = 2,3$ and propose a conjecture for general κ.

## 2. Definitions and Preliminaries

**Cycle Graph**: A graph $C_{\xi}$ is a cycle with nn vertices connected in a closed chain.
**Graph Power**: The $\kappa^{th}$ power of a graph G, denoted $G^{\kappa}$, is a graph on the same vertex set where two vertices are adjacent if their distance in G is at most κ.
**Splitting Graph**: Given a graph $G(V, E)$, the splitting graph $Sp(G)$ is obtained by adding a vertex for each edge $e \in E(G)$, and joining it to the endpoints of e.
**Dominating Set**: A subset $D \subseteq V(G)$ is a dominating set if every vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D.
**Domination Number**: The domination number γ(G) is the minimum cardinality of a dominating set in G.

## 3. Main Results

In this section, we investigate the domination number of the splitting graph of the higher powers of cycle graphs. Our study is motivated by a result due to Vaidya et al. [13], who determined the domination number of the splitting graph of a standard cycle graph $C_n$. We aim to extend their findings by computing the domination number of the splitting graph of $C_{\xi}^{\kappa}$ for $\kappa = 2,3$, and by proposing a conjecture for general κ. These results highlight how the structure of a graph changes under the power and splitting operations, and how these changes affect domination parameters.
**Theorem 3.1** (Vaidya et al. [13])
Let $C_\xi$ be a cycle graph with $\xi \geq 3$. Then the domination number of the splitting graph $Sp(C_\xi)$ is given by

$$\gamma(Sp(C_\xi))) = \begin{cases} \frac{\xi}{2} & \text{for } \xi \equiv 0 \pmod 4 \\ \frac{\xi+2}{2} & \text{for } \xi \equiv 2 \pmod 4 \\ \frac{\xi+1}{2} & \text{for } \xi \equiv 1,3 \pmod 4 \end{cases}$$

Building upon this, we now consider the splitting graphs of $C_{\xi}^{2}$ and $C_{\xi}^{3}$. Our goal is to determine their domination numbers and to generalize the pattern observed.

**Theorem 3.2** For $\xi \geq 7$,

$$\gamma\left(Sp(C_\xi^2)\right) = \begin{cases} \frac{2\xi}{7} & \text{for } \xi \equiv 0 \bmod(7) \\ \frac{2\xi+5}{7} & \text{for } \xi \equiv 1 \bmod(7) \\ \frac{2\xi+3}{7} & \text{for } \xi \equiv 2 \bmod(7) \\ \frac{2\xi+8}{7} & \text{for } \xi \equiv 3 \bmod(7) \\ \frac{2\xi+6}{7} & \text{for } \xi \equiv 4 \bmod(7) \\ \frac{2\xi+4}{7} & \text{for } \xi \equiv 5 \bmod(7) \\ \frac{2\xi+2}{7} & \text{for } \xi \equiv 6 \bmod(7) \end{cases}$$

**Proof**

Let $\omega_1, \omega_2, \omega_3, \dots, \omega_\xi$ are vertices of $C_\xi^2$ and $\omega_1', \omega_2', \omega_3', \dots, \omega_\xi'$ are the corresponding vertices to $\omega_1, \omega_2, \omega_3, \dots, \omega_\xi$, which are added to form $Sp(C_\xi^2)$. Since $N(\omega_i) = \{\omega_{i-2}, \omega_{i-1}, \omega_{i+1}, \omega_{i+2}, \omega'_{i-2}, \omega'_{i-1}, \omega'_{i+1}\omega'_{i+2}$ and $N(\omega_i') = \{\omega_{i-2}, \omega_{i-1}, \omega_{i+1}, \omega_{i+2}\}$, so atleast one vertex from $\omega_{i-2}, \omega_{i-1}, \omega_{i+1}$ and $\omega_{i+2}$ must belong to any dominating set of $p(C_\xi^2)$. Consequently $|S| \geq \frac{2\xi}{7}$. Now consider the following subsets for $0 \leq j < \lfloor \frac{\xi}{7} \rfloor$:

$S = \{\omega_{3+7i}, \omega_{5+7i}\}$, for $\xi \equiv 0 \mod\{7\}$ such that $|S| = \frac{2\xi}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-2}\}$, for $\xi \equiv 1 \mod\{7\}$ such that $|S| = \frac{2(\xi-1)}{7} + 1 = \frac{2\xi+5}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-2}\}$, for $\xi \equiv 2 \mod\{7\}$ such that $|S| = \frac{2(\xi-2)}{7} + 1 = \frac{2\xi+3}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-4}, \omega_{\xi-2}\}$, for $\xi \equiv 3 \mod\{7\}$ such that $|S| = \frac{2(\xi-3)}{7} + 2 = \frac{2\xi+8}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-4}, \omega_{\xi-2}\}$, for $\xi \equiv 4 \mod\{7\}$ such that $|S| = \frac{2(\xi-4)}{7} + 2 = \frac{2\xi+6}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-4}, \omega_{\xi-2}\}$, for $\xi \equiv 5 \mod\{7\}$ such that $|S| = \frac{2(\xi-5)}{7} + 2 = \frac{2\xi+4}{7}$.

$S = \{\omega_{3+7i}, \omega_{5+7i}, \omega_{\xi-4}, \omega_{\xi-2}\}$, for $\xi \equiv 6 \mod\{7\}$ such that $|S| = \frac{2(\xi-6)}{7} + 2 = \frac{2\xi+2}{7}$.

We propose that S forms a dominating set, since $N(\omega_{3+7i}) = \{\omega_{1+7i}, \omega_{2+7i}, \omega_{4+7i}, \omega_{5+7i}, \omega'_{1+7i}, \omega'_{2+7i}, \omega'_{4+7i}, \omega'_{5+7i}\}$ and

$N(\omega_{5+7i})=\{\omega_{3+7i}, \omega_{4+7i}, \omega_{6+7i}, \omega_{7+7i}, \omega'_{3+7i}, \omega'_{4+7i}, \omega'_{6+7i}, \omega'_{7+7i}\}$ such that $\lfloor\frac{2\xi}{5}\rfloor$ vertices are dominated by vertex of the form $N(\omega_{3+7i})$ and $N(\omega_{5+7i})$the rest are dominated by $\omega_{\xi-2}$ , for $\xi \equiv 1,2 \mod (7)$ and by $\omega_{\xi-2}$ or $\omega_{\xi-4}$, for $\xi \equiv 3,4,5,6 \mod (7)$. Also each $S$ is a minimal dominating set because by removing the vertex $\omega_{5+7i}$, the vertex $\omega'_{3+7i}$ will not be dominated by any of the vertices. Hence,

$$\gamma\left(Sp\left(C_{\xi}^{2}\right)\right) = \begin{cases} \frac{2\xi}{7} & \text{for } \xi \equiv 0 \bmod(7) \\ \frac{2\xi+5}{7} & \text{for } \xi \equiv 1 \bmod(7) \\ \frac{2\xi+3}{7} & \text{for } \xi \equiv 2 \bmod(7) \\ \frac{2\xi+8}{7} & \text{for } \xi \equiv 3 \bmod(7) \\ \frac{2\xi+6}{7} & \text{for } \xi \equiv 4 \bmod(7) \\ \frac{2\xi+4}{7} & \text{for } \xi \equiv 5 \bmod(7) \\ \frac{2\xi+2}{7} & \text{for } \xi \equiv 6 \bmod(7) \end{cases}$$

**Theorem 3.3** For $\xi \geq 10$,

$$\gamma\left(Sp\left(C_{\xi}^{3}\right)\right) = \begin{cases} \frac{\xi}{5} & \text{for } \xi \equiv 0 \bmod(10) \\ \frac{\xi+4}{5} & \text{for } \xi \equiv 1, 6 \bmod(10) \\ \frac{\xi+3}{5} & \text{for } \xi \equiv 2, 7 \bmod(10) \\ \frac{\xi+2}{5} & \text{for } \xi \equiv 3, 8 \bmod(10) \\ \frac{\xi+6}{5} & \text{for } \xi \equiv 4 \bmod(10) \\ \frac{\xi+5}{5} & \text{for } \xi \equiv 5 \bmod(10) \\ \frac{\xi+1}{5} & \text{for } \xi \equiv 9 \bmod(10) \end{cases}$$

**Proof**

Let $\omega_1, \omega_2, \omega_3, \dots, \omega_\xi$ are vertices of $C_\xi^3$ and $\omega_1', \omega_2', \omega_3', \dots, \omega_\xi'$ are the corresponding vertices to $\omega_1, \omega_2, \omega_3, \dots, \omega_\xi$, which are added to form $Sp(C_\xi^3)$. Since $N(\omega_i) = \{\omega_{i-3}, \omega_{i-2}, \omega_{i-1}, \omega_{i+1}, \omega_{i+2}, \omega_{i+3}, \omega_{i-3}', \omega_{i-2}', \omega_{i-1}', \omega_{i+1}'\omega_{i+2}'\omega'_{i+3},\}$ and $N(\omega_i') = \{\omega_{i-3}, \omega_{i-2}, \omega_{i-1}, \omega_{i+1}, \omega_{i+2}, \omega_{i+3}\}$, so atleast one vertex from $\omega_{i-3}, \omega_{i-2}, \omega_{i-1}, \omega_{i+1}, \omega_{i+2}$ and $\omega_{i+3}$ must belong to any dominating set of $Sp(C_\xi^2)$.. Consequently $|S| \geq \frac{2\xi}{7}$. Now consider the following subsets for $0 \leq j < \lfloor \frac{\xi}{10} \rfloor$:

$S = \{\omega_{4+10i}, \omega_{7+10i}\}$, for $\xi \equiv 0 \mod (10)$ such that $|S| = \frac{\xi}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-3}\}$, for $\xi \equiv 1 \mod (10)$ such that $|S| = \frac{2(\xi-1)}{10} + 1 = \frac{\xi+4}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-3}\}$, for $\xi \equiv 2 \mod (10)$ such that $|S| = \frac{2(\xi-2)}{10} + 1 = \frac{\xi+3}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-3}\}$, for $\xi \equiv 3 \mod (10)$ such that $|S| = \frac{2(\xi-3)}{10} + 1 = \frac{\xi+2}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 4 \mod (10)$ such that $|S| = \frac{2(\xi-4)}{10} + 2 = \frac{\xi+6}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 5 \mod (10)$ such that $|S| = \frac{2(\xi-5)}{10} + 2 = \frac{\xi+5}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 6 \mod (10)$ such that $|S| = \frac{2(\xi-6)}{10} + 2 = \frac{\xi+4}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 7 \mod (10)$ such that $|S| = \frac{2(\xi-7)}{10} + 2 = \frac{\xi+3}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 8 \mod (10)$ such that $|S| = \frac{2(\xi-8)}{10} + 2 = \frac{\xi+2}{5}$.

$S = \{\omega_{4+10i}, \omega_{7+10i}, \omega_{\xi-6}, \omega_{\xi-3}\}$, for $\xi \equiv 9 \mod (10)$ such that $|S| = \frac{2(\xi-9)}{10} + 2 = \frac{\xi+1}{5}$.

We propose that S forms a dominating set, since $N(\omega_{4+10i}) = \{\omega_{1+10i}, \omega_{2+10i}, \omega_{3+10i}, \omega_{5+10i}, \omega_{6+10i}, \omega_{7+10i}, \omega'_{1+10i}, \omega'_{2+10i}, \omega'_{3+10i}, \omega'_{5+10i}, \omega'_{6+10i}, \omega'_{7+10i}$ and $N(\omega_{7+10i}) = \{\omega_{4+10i}, \omega_{5+10i}, \omega_{6+10i}, \omega_{8+10i}, \omega_{9+10i}, \omega_{10+10i}, \omega'_{4+10i}, \omega'_{5+10i}, \omega'_{6+10i}, \omega'_{8+10i}, \omega'_{9+10i}, \omega'_{10+10i}$ such that $\lfloor \frac{\xi}{5} \rfloor$ vertices are dominated by vertex of the form $N(\omega_{4+10i})$ or $N(\omega_{7+10i})$ the rest are dominated by $\omega_{\xi-3}$ , for $\xi \equiv 1,2,3 \mod (10)$ and by $\omega_{\xi-3}$ or $\omega_{\xi-6}$, for $\xi \equiv 4,5,6,7,8,9 \mod (10)$. Also each $S$ is a minimal dominating set because by removing the vertex $\omega_{7+10i}$, the vertex $\omega'_{4+10i}$ will not be dominated by any of the vertices. Hence,

$$\gamma\left(Sp\left(C_{\xi}^{3}\right)\right)=\begin{cases}\frac{\xi}{5} & for\ \xi\equiv 0\ mod\ (10)\\ \frac{\xi+4}{5} & for\ \xi\equiv 1,6\ mod\ (10)\\ \frac{\xi+3}{5} & for\ \xi\equiv 2,7\ mod\ (10)\\ \frac{\xi+2}{5} & for\ \xi\equiv 3,8\ mod\ (10)\\ \frac{\xi+6}{5} & for\ \xi\equiv 4\ mod\ (10)\\ \frac{\xi+5}{5} & for\ \xi\equiv 5\ mod\ (10)\\ \frac{\xi+1}{5} & for\ \xi\equiv 9\ mod\ (10)\end{cases}$$

**Conjecture 3.1** For $\xi \geq 3\kappa + 1$,

$$\gamma\left(Sp\left(C_{\xi}^{\kappa}\right)\right)=\begin{cases}\frac{2\xi}{3\kappa+1} & \text{for } \xi\equiv 0 \text{ mod } (3\kappa+1)\\ \frac{2(\xi-\mathrm{i})}{3\kappa+1}+1 & \text{for } \xi\equiv 1,2,3,\dots,\kappa \text{ mod } (3\kappa+1)\\ \frac{2(\xi-\mathrm{i})}{3\kappa+1}+2 & \text{for } \xi\equiv \kappa+1,\kappa+2,\kappa+3,\dots,3\kappa \text{ mod } (3\kappa+1)\end{cases}$$

## 4. Conclusion

In this work, we focused on computing the domination number of the splitting graph of higher powers of cycle graphs. Through casewise analysis, we found exact values for $\kappa = 2,3$. Our computations reveal structural patterns, which led us to propose a conjecture for the general $\kappa^{\text{th}}$ power. This study contributes to the theory of domination in transformed graphs and opens avenues for further research, including tighter bounds, algorithmic techniques, and generalizations to other graph families.

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