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On Partition Dimension of Some Class of Rotationally Symmetric Graphs.

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Abstract

For a simple connected graph G, the distance $d(a_i, b_i)$, where $a_i, b_i \in V(G)$, is the length of the shortest path, measured by the number of edges, between vertices a_i and b_i. The n-order partition of vertices of G is denoted as $\psi = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_n\}$. The notation of vertex $a_i \in G$ with respect to ψ is the vector code $\{d(a_i, \psi_1), d(a_i, \psi_2), \ldots, d(a_i, \psi_n)\}$. Partition set ψ is called a resolving partition set if the representation of each vertex with respect to ψ is unique. The partition dimension of G is defined as the minimum size of such a resolving partition set. In this research, we investigated the partition dimension of the generalized gear graph G(k, n) and the generalized fan graph F (2, n)..

Key Words: Partition dimension, Metric dimension, Resolving set, Resolving partition set, Gear graph and Fan graph.

Introduction

In a simple connected graph G with a finite set of vertices V and edges E, the distance $d(a_i, U)$ from a vertex $a_i \in V$ to a subset $U \subseteq V$ is defined as the minimum distance $\min\{d(a_i, u_j) \mid u_j \in U\}$. For a subset $U = \{u_1, u_2, \ldots, u_k\}$, the representation $r(a_i \mid U)$ of a vertex a_i with respect to U is the ordered tuple $(d(a_i, u_1), d(a_i, u_2), \ldots, d(a_i, u_k))$. A subset U is termed a resolving set of G if, for any distinct vertices $a_i, b_i \in V$, the representations $r(a_i \mid U) \neq r(b_i \mid U)$. The metric dimension of G is the cardinality of the smallest such resolving set. From 1975 onward, the notion of metric dimension and metric basis is described in academic sources under various names. Slater called it locating set in [1] and Harary et al. in 1976 called basis set [2]. The metric dimension of graphs has various uses in computer science and optimization. Chatrand et al. in 1998 extended this concept to partition dimension [3]. A partition of a set is a group of non-overlapping subsets whose union encompasses the entire set. The partition dimension pertains to partitioning the vertex set V of a graph G, emphasizing the graph's resolvability, where vertices are uniquely identified by their distances to the subsets within the partition set. Let $\psi = \{\psi_1, \psi_2, \psi_3, \ldots, \psi_\mu\}$ be an $\mu-$ partition of V (G), the notation of vertex $a_i \in V$ in reference to ψ, is the vector code $(d(a_i, \psi_1), d(a_i, \psi_2), d(a_i, \psi_3), \ldots, d(a_i, \psi_\mu))$. If $r(a_i|\psi) \neq r(b_i|\psi)$ $\forall a_i, b_i \in V(G)$ and $a_i \neq b_i$ then the partition set ψ is termed as resolving partition set of G and if their is no other partition resolving set of G which has least order than ψ then the cardinality of ψ is called partition dimension. Chatrand et al. in 2000 found an important result that Pdim(G) ≤ dim(G) + 1 [4]. For further detail about metric dimension and partition dimension of graphs, we refer [5,6].In 2020, E. T. Baskoro and colleagues determined that graphs with order $\xi \geq 11$ and diameter 2 possess a partition dimension of $\xi - 3$ [7]. In 2023, Shah and co-authors explored the parti- tion dimension of generalized convex polytopes [8]. The discovery of the fullerene molecule by Kroto et al. in 1985 sparked significant research interest in fullerene graphs [9]. In 1975, Garey and colleagues demonstrated that identifying a graph's resolving set is an NP-hard problem [11]. Z. Hussain et al. in 2019 established upper bounds for the partition dimension of the M-wheel graph [12]. In 2020, M. Azeem and colleagues derived precise bounds for the partition dimension of convex poly- topes [13]. C. Grigorious et al. in 2017 and Maritz et al. in 2018 investigated the partition dimension of circulant graphs [14, 15]. G. Chappel and colleagues exam- ined the relationships among diameter, partition dimension, metric dimension, and other graph properties [17].In 2010, I. G. Yero and co-authors explored bounds on the partition dimension of Cartesian product graphs [18]. They proved that for any connected graphs A and B, the partition dimension of their Cartesian product satisfies Pd(A × B) ≤ Pd(A) + Pd(B) and Pd(A × B) ≤ Pd(A) + dim(B). These results clarify how partition and metric dimensions interact in product graphs, aiding applications in network design and graph algorithms. I. Javaid et al. in 2008 derived partition dimension of graphs related to wheels [19]. A. Khalil et al. in 2022, determined the partition dimension for specific classes of convex polytopes containing pendant edges and proved that its partition dimension is finite [20]. A. Nadeem et al. in 2022 calculate the Partition dimensions of notable convex polytope families. They gave bounds for different graph properties related to metric and partition dimensions and provided general examples of graphs with specific partition dimensions in [21].

H. Raza et al. in 2021, assessed the cardinalities of the generalized Petersen graph to determine the upper bound for its partition dimension [22]. M. A. Mohammed et al. in 2021, examined the partition dimension of chain cycles formed by even and odd cycles [23]. Their research primarily focused on the partition dimension of the planar tessellation derived from boron nanosheets. Furthermore, they also considered a few induced subgraphs of these sheets to study their metric dimension. The applications of partition dimension in different fields like hierarchical data structures, robot navigation, network verification, network discovery, chemical compound representation, master mind game strategies, Djokovic–Winkler relation, image processing and identification of patterns.

The partition dimension of Gear graphs and Fan graph has a broad range of applications in network design, optimization, and systems analysis. The partition di- mension of gear graphs assists in creating efficient routing algorithms and optimizing communication networks, error detection and correction in distributed systems, as well as recognizing communities in social networks. On the other hand, the fan graph finds applications in graph drawing and visualization by simplifying complex cyclic structures, circuit design by modeling feedback loops, network topology for resilient routing, and combinatorial optimization in problems like the traveling salesman prob- lem. Moreover, these graphs are useful in dynamic systems modeling, particularly for analyzing feedback processes in control systems and biological networks. These con- cepts provide powerful tools for understanding and optimizing various real world systems.

The partition dimension of a gear graph, which features a central vertex linked to an even cycle with pendant vertices attached to alternating cycle nodes, is a key tool in optimization across various domains. In network optimization, gear graphs model hub-and-spoke systems like telecommunications or transportation networks, where the central vertex acts as a core server or hub, the cycle represents regional nodes, and pendant vertices denote endpoints like user devices or delivery points; the partition dimension minimizes the number of landmark nodes needed for efficient routing, reducing latency and costs in systems like 5G networks or airline routes. In logistics, gear graphs represent supply chains with a central warehouse connected to cyclic regional hubs and local delivery points, where the partition dimension optimizes the selection of key distribution centers to cover all endpoints, cutting transportation costs for platforms like e-commerce. In social network analysis, gear graphs model a central influencer linked to a cyclic community with sub-groups, and the partition dimension identifies minimal influential nodes to maximize marketing reach, enhancing campaign efficiency. For error detection in sensor networks, the partition dimension selects minimal monitoring nodes to detect failures, improving reliability in systems like environmental monitoring. In combinatorial optimization, such as scheduling in distributed computing, it optimizes task assignments by pinpointing critical nodes, streamlining cloud computing operations. Despite challenges like computational com- plexity, future algorithmic improvements will enhance the partition dimensions utility in optimizing complex systems. For more applications of partition dimension of graphs, we refer [2, 5-8]. Since there are many graphs whose partition dimension is unknown therefore its applications are still limited. As a result, partition dimension is among the interesting problems to solve in graph theory. The following findings are helpful in calculating the partition dimension of our graphs.

Theorem 0.1. [1] Let ϕ be a resolving partition set of V(G) , if $d(v_i, k_i) = d(v_j, k_i)$ for all $k_i \in V(G) - \{v_i, v_j\}$, then v_i and v_j must be in different classes of ϕ.

Theorem 0.2. [8] For S_ω a convex polytope graph with $\omega \geq 6$ then $Pd(S_\omega) \leq 4$.

Theorem 0.3. [19] For a class of circulant graphs $G_\alpha(1, 3)$ if $\alpha \equiv 1 (\mathrm{mod}\ 6)$ and $\alpha \geq 13$ then $Pdim(G_\alpha) \leq 4$.

Main Results Generalized Gear graph:

The generalized gear graph G(k, n) is constructed by modifying a wheel graph where n additional vertices are inserted between each adjacent pair of cycle vertices. In a generalized gear graph G(k, n), the vertex of degree k is labeled by u_0, called gear graph's center. Neighbors of u_0 are labeled with $u_1, u_2, ..., u_k$ and n vertices inserted between every two neighboring vertices of the central vertex.

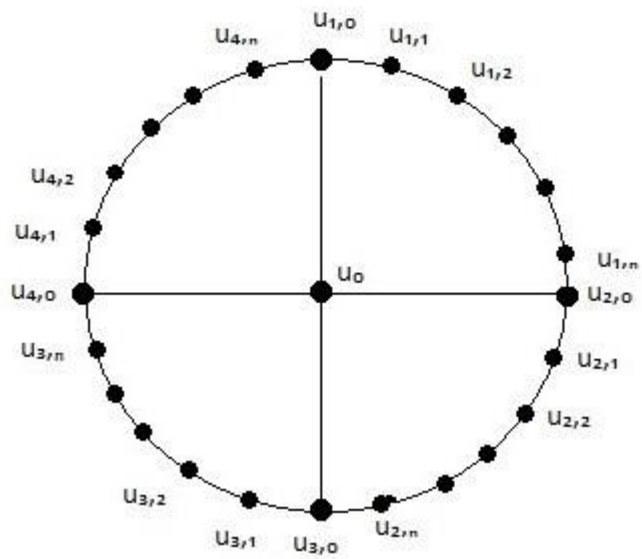

Figure 1: GENERALIZED GEAR GRAPH

Theorem 0.4. Let $G_{k,n}$ be a generalized gear graph then $Pdim(G_{k,n}) \leq 4$ for $n \geq 3$.
Proof. Here we have two cases.
Case-I: For odd vertices i.e $n = 2\alpha - 1$, $\alpha = 2, 3, 4, ..., n$, we partitioned the ver- tices into four partition resolving sets $\omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with $\omega_1 = \{u_0, u_{1,\mu}\}$, $\omega_2 = \{u_{2,\mu}\}$, $\omega_3 = \{u_{3,\mu}\}$ and $\omega_4 = \{u_{4,\mu}\}$ where $0 \leq \mu \leq n$. To achieve the desired result, it is enough to demonstrate that each vertex in $G_{k,n}$ has a distinct representation concerning ω.

The representation of vertices will be of the following form

$$\mathbf{r(u_{1,\mu} \mid \omega)} = \begin{cases} (0, \mu + 2, \mu + 2, \mu + 1) \text{ if } 0 \leq \mu \leq \alpha - 1 \\ \left(0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 6 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, \mu + 1\right) \text{ if } \mu = \alpha \\ 0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 6 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor \text{ if } \alpha + 1 \leq \mu \leq n \end{cases}$$

$$\mathbf{r(u_{2,\mu} \mid \omega)} = \begin{cases} (\mu, 0, \mu + 2, \mu + 3) \text{ if } 0 \leq \mu \leq \alpha - 1 \\ \left(\mu, 0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) \text{ if } \mu = \alpha \\ \left(5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) \text{ if } \alpha + 1 \leq \mu \leq n \end{cases}$$

$$\mathbf{r(u_{3,\mu} \mid \omega)} = \begin{cases} (\mu + 1, \mu, 0, \mu + 3) \quad \text{if} \quad 0 \leq \mu \leq \alpha - 1 \\ \left(\mu + 1, \mu, 0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) \text{ if } \mu = \alpha \\ \left(5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 6 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0{,}5 - \mu + 2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) \text{ if } \alpha + 1 \leq \mu \leq n \end{cases}$$

$$r(u_{4,\mu} \mid \omega = \begin{cases} (\mu+1, \mu+2, \mu, 0) & \text{if} \quad 0 \le \mu \le \alpha-1 \\ \left(4-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, \mu+2, \mu, 0\right) & \text{if } \mu = \alpha \\ \left(4-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0\right) & \text{if } \alpha+1 \le \mu \le n \end{cases}$$

We can easily verify that all vertices have distinct representation with respect to resolving partition ω. It shows that When n is odd, we can divide all of the vertices into four partition resolving sets.

Case-II: For even number of vertices $n = 2\alpha$ and $2 \le \alpha \le n$ the representation of vertices of circle is of the form

$$r(u_{1,\mu} \mid \omega) = \begin{cases} (0, \mu+2, \mu+2, \mu+1) & \text{if } 0 \le \mu \le \alpha \\ \left(0, \mu, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, \mu+1\right) & \text{if } \mu = \alpha+1 \\ 0, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 8-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor & \text{if } \alpha+2 \le \mu \le n \end{cases}$$

$$r(u_{2,\mu} \mid \omega) = \begin{cases} (\mu, 0, \mu+2, \mu+3) & \text{if } 0 \le \mu \le \alpha \\ \left(6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 8-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) & \text{if } \alpha+1 \le \mu \le n \end{cases}$$

$$r(u_{3,\mu} \mid \omega) = \begin{cases} (\mu+1, \mu, 0, \mu+3) & \text{if} \quad 0 \le \mu \le \alpha-1 \\ \left(\alpha+1, \mu, 0, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) & \text{if } \alpha \le \mu \le \alpha+1 \\ \left(6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0, 6-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor\right) & \text{if } \alpha+2 \le \mu \le n \end{cases}$$

$$r(u_{4,\mu} \mid \omega = \begin{cases} (\mu+1, \ \mu+2, \ \mu, \ 0) & \text{if} \quad 0 \le \mu \le \alpha-1 \\ \left(5-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, \mu, 0\right) & \text{if } \mu = \alpha+\mathbf{1} \\ \left(5-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 7-\mu+2\left\lfloor\frac{\kappa}{2}\right\rfloor, 0\right) & \text{if } \alpha+2 \le \mu \le n \end{cases}$$

We can check that each vertex has a unique representation with respect to the par- tition set ω. It demonstrates that When n is even, we can divide all of the vertices into four partition sets. In both cases, no two vertices have the same representation and hence $Pdim \le 4$.

□

1GENERALIZED FAN GRAPH:

The fan graph F (2, n) is formed by taking the graph join of an empty graph $\bar{K}_n$ with n vertices (no edges) and a path graph P_2 with two vertices (one edge). The vertex set of F (2, n) consists of all vertices from $\bar{K}_n$ and P_2 combined, and its edge set includes the single edge of P_2 plus all edges connecting each vertex of $\bar{K}_n$ to both vertices of P_2.. Clearly the total number of vertices of F (2, n) are n + 2 and total number edges are 2m + 1.

Theorem 1.1. If F (2, n) is a Fan graph where $n \geq 2$ then Pdim(F (2, n)) = n + 2.

Proof. Let the partition set of F (2, n) be $W = \{w_1, w_2, w_3, ..., w_n\}$ with $w_1 = \{u_1\}$, $w_2 = \{u_2\}$, $w_3 = \{u_3\}$,..., $w_n = \{u_n\}$ then the notation of nodes of F (2, n) with respect to W will be of the form;

$$r(u_1|W) = (0, 1, 1, 1, \overset{(n-1)}{\overbrace{\text{ti.mes}}}, 1)$$
$$r(u_2|W) = (1, 0, 1, 1, \overset{(n-2)}{\overbrace{\text{ti.m..es}}}, 1)$$
$$r(u_3|W) = (1, 1, 0, 2, 2, 2, \overset{(n-3)}{\overbrace{\text{ti.m..es}}}, 2)$$
$$r(u_4|W) = (1, 1, 2, 0, 2, 2, \overset{(n-4)}{\overbrace{\text{ti.m..es}}}, 2)$$
$$.$$
$$r(u_{n-1}|W) = (1, 1, 2, 2, 2, \overset{(n-4)}{\overbrace{\text{ti.m..es}}}, 2, 0, 2)$$
$$r(u_n|W) = (1, 1, 2, 2, 2, \overset{(n-3)}{\overbrace{\text{ti.m..es}}}, 2, 0)$$

it is noted that no two vertices have same representation so Pdim(F (2, n)) $\leq$ n + 2. Next we will show that Pdim(F (2, n)) $\geq$ n + 2 by providing that their is no resolving partition set W' such that $|W'| \leq |W|$. On the contrary let W' be a resolving partition of F (2, n) such that $|W'| \leq |W|$ then $r(u_{n-1}|W) = r(u_n|W) = (1, 1, 2, 0)$ which is a contradiction. This contradiction is due to our wrong supposition and hence we conclude that Pdim(F (2, n)) = n + 2.

□

Conclusion

In this research paper, we conducted a comprehensive investigation into the partition dimension of two distinct classes of graphs: the generalized gear graph, denoted as G(k, n), and the fan graph, denoted as F (2, n). Our analysis reveals significant insights into the structural properties of these graphs. Specifically, we established that the partition dimension of the generalized gear graph G(k, n) is bounded above by 4, indicating a compact resolving partition for this graph family. In contrast, our findings demonstrate that the partition dimension of the fan graph F (2, n) is precisely n + 2, reflecting a linear dependency on the parameter n. These results contribute to a deeper understanding of the combinatorial properties and resolving capabilities of these graph structures, with potential implications for their applications in network design, graph theory, and related fields.

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