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Bifurcation Phenomena and Dynamic Behavior of Periodic Solutions in First-Order Non-Autonomous Differential Equations

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Abstract

We consider equations of the form

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

Where α and β are polynomial functions of t with a real dependent variable, but z is complex such equations were considered by Lins Neto [7]. Our particular interest is the maximum number of periodic solutions which can bifurcate out of the origin following [1] and[3] , we consider different classes of equations $C_{11,11}$, $C_{18,1}$ and $C_{18,2}$ of the form (4) and we will calculate the maximum possible multiplicity of the origin using theorem [9]. We use Maple to calculate focal values of, $C_{18,1}$, $C_{18,2}$ and $C_{18,3}$.

Inauguration

Preface

In nature, many phenomena repeat after some time, such type of phenomena are mathematically defined as periodic solutions of system and generally we consider system consisting of differential equations. In 20th century investigation about periodic solution was a hot topic of research and take an important part to development of modern mathematics. In almost all fields of science, research on this topic has an impact for development of that field. On August 8, 1900, David Hilbert conveyed 23 crucial mathematical issues at the Second International Congress of Mathematics [1]. Hilbert Problems gives further directions of Mathematics. 16th problem is most manipulated problem of algebraic geometry, differential equations and topology. 16th problem having two parts. In the first part we discuss relative position of algebraic curves in higher order vector field. In this part we find relation between different geometric properties of curves to understand the basic actions of complex systems. In second part, he discusses existence of possible number of limit cycle and it's in higher dimensional polynomial vector fields their relative position. Limit cycles guide us to understand the behavior of dynamical systems. This problem has extensive impact to understand dynamical systems and ordinary differential equations. Tracking number of and location of limit cycles is essential in recognize the behavior of complex systems of population dynamics, chemical reactions, and electrical circuits. Limit cycle theory is told us about existence, uniqueness, and stability of limit cycles which helps us to recognize the behavior of nonlinear systems [2]. This theory also helps in development of physics, engineering, and biology. Hilbert's 16th problem has had a profound impact on the development of mathematics and science. The problem has inspired generations of mathematicians and scientists, leading to significant advances in our understanding of complex systems. The active aera of research is about more deep knowledge of limit cycles and its properties. Finding limit cycles of differential equations is complicated. Henri Poincaré find and talk about limit cycle in his article first time having

4 parts “Integral Curves Described by Differential Equations” [6-9], published after 1880. Poincaré identify strong relation of the behaviours of limit cycles and solutions of differential equation problems related to structural of integral curves. After above research, Ivar Bendixson expand Poincaré’s work and this is known as Poincaré-Bendixson theorem [3], which tells in bounded area, limit set number of paths of dynamical systems. The triode vacuum tube in which self-produced oscillations are generated is invented as a result of limit cycle theory. Linear differential equations were found to explain such type of oscillation. In late 1920s, Van der Pol find a differential equation to explain oscillations for a triode vacuum tube of constant amplitude. [4]. In dynamical systems self-excited oscillations gives solutions. The Holing-Tanner predator-prey model [5], heart rhythms, temperature rhythms of body, hormone secretion, chemical reactions that are spontaneously oscillating, and vibrations in bridges and wings of airplane are examples of sustained oscillations even without presence of external effect’

Consider the differential equation of type

$dz/dț = r_0(ț)\, z^3 + r_1(ț)\, z^2 + r_2(ț)\, z + r_3(ț)\ldots\ldots$ (1.1.1)

Is considered, ʐ is complex function depending on t, r_i’s are functions that are really valued. This form is same as equation given below:

$dz/dț = q_0(ț) z_n + q_1(ț) z_{n-1} + \ldots + q_n(ț)$ (1.1.2)

Problem that is discussing here is about obtaining largest possible limit cycles of system that can bifurcate from origin or we can say it as to find the periodic solutions that generated from ʐ = 0 by using perturbation of the coefficient of equation (1.1.1).Many phenomena are represented by system of differential equations ,where solution of system repeat with period of time. Most of research was on specific differential equations representing mechanics and electronics model ,but latest research is application to fields of Biology ,demography and economics for this research new differential equations are introduced that are periodic and dependent of time ,so topic of interest is periodic solution of non-autonomous differential equation also finding maximum number of periodic solution is most important topic of research.

Some basic definition used in the study of limit cycles are

Explanations:

Autonomous equation

A differential equation with right hand side having no independent variable

$$\frac{ds}{dț} = ʄ(s) \qquad (1.2.1)$$

is called autonomous equation.

Non-autonomous equation

A differential equation in which independent variable exists explicitly on right hand side of equation

$$\frac{ds}{dț} = ʄ(ț, s) \qquad (1.2.2)$$

is non-autonomous equation.

Explicit function

A function in which dependent variable can be written in the form of function of independent variable.

$s = ʄ(ț)$ (1.2.3)

Implicit function

A function in which dependent variable is not equal to a function of independent variable.

$ʄ(s, ț) = 0$ (1.2.4)

Periodic solution

If $s(ț)$ is any solution of

$$\frac{ds}{dț} = ʄ(s) \qquad (1.2.5)$$

if

$$s(ȶ + w) = ʂ(ȶ) \tag{1.2.6}$$

where w is lowest positive number.

Critical point

Q $(s_0, ȶ_0)$ is critical point of given below system

$$\frac{ds}{dȶ} = X(s, ȶ)$$
$$\frac{ds}{dȶ} = ℽ(s, ȶ)$$

if

$$X\ (s_0, ȶ_0) = 0,$$

and

$$ℽ(x_0, y_0) = 0.$$

Center

A center is a critical point where it is closed by many closed paths.

Limit cycle

From all periodic orbits of planar differential system, an isolated periodic orbit is called limit cycle.

1.3 Polynomial system and Abel's equation

Periodic solutions of equation (1.1.1) is our main concern. Consider the system

$$\frac{dꝀ}{dȶ} = ⋋\, Ꝁ + Ꜧ + \Phi_{ɭ}(Ꝁ, Ꜧ) \tag{1.3.1}$$
$$\frac{dꜦ}{dȶ} = -Ꝁ + ⋋\, Ꜧ + \Psi_{ɭ}(Ꝁ, Ꜧ)$$

Φ_n and Ψ_n are homogeneous polynomials of $ɭ$.

Consider transformation

$$\mathfrak{J}: (Ɽ, ꝗ) \longrightarrow (ꝥ, ꝗ) \tag{1.3.2}$$

where

$$ꝥ = Ɽ^{ɭ-1}\left(1 - Ɽ^{ɭ-1} g(ꝗ)\right)^{-1} \tag{1.3.3}$$

In an open set $\mathfrak{D} = \{(Ɽ, ꝗ), Ɽ^{ɭ-1} g(ꝗ) < 1\}$ containing the origin, above function is defined.

This transformation in polar coordinates system has been used for investigating periodic solutions

$$\frac{dⱤ}{dȶ} = ⋋\, Ɽ + Ɽ^{ɭ}\, ƒ(ꝗ), \frac{dꝗ}{dȶ} = 1 - Ɽ^{(ɭ+1)}(g(ꝗ)) \tag{1.3.4}$$

$ƒ(ꝗ), g(ꝗ)$ are polynomials of degree ꝗ + 1 in cosꝗ and sinꝗ.

When $ɭ = 2$ the above discussed transformation was explained firstly by Lins Neto [11].

From (1.3.3), we get

$$\frac{dꝥ}{dt} = \alpha(t)ꝥ^3 + \beta tꝥ^2 - ⋋\,(ɭ - 1)ꝥ \tag{1.3.5}$$

where

$$\alpha(t) = -(ɭ - 1)g(t)\{ƒ(t) + ⋋\, g(t)\} \tag{1.3.6}$$
$$\beta(ꝗ) = -(ɭ - 1)\{ƒ(t) + 2 ⋋\, g(t) + g(t)\}$$

From (1.3.6) , the transformation (1.3.3) can be written as

$$Ɽ^{ɭ-1} = \frac{ꝥ}{1 + ꝥ g(ꝗ)} \tag{1.3.7}$$

Transformation maps from region $Ɽ = 0$ to $ꝥ = 0$ *and* $Ɽ > 0$ to $ꝥ > 0$,also from neighboring region of $Ɽ = 0$ to neighborhood of $ꝥ = 0$, if $ꝥ > 0$ with $1 + ꝥ g(ꝗ) > 0\ \forall\, \Theta$. It can be verify that the system given in (1.3.1),of constant solution $ꝥ = 0$ related to equilibrium point of (1.3.5) and also periodic solution of (1.3.5) having small positive $ꝥ$ with low frequency limit cycles of (1.3.1) .

Consider

$$d\boldsymbol{x}/d\mathfrak{t} = \mathbf{A}(\boldsymbol{x}, \boldsymbol{y})$$
$$d\boldsymbol{y}/d\mathfrak{t} = \mathbf{E}(\boldsymbol{x}, \boldsymbol{y})$$

A and **E** are **m**th-degree polynomials.
Let π(k,H) specify limit cycles oof above system. We clarify it as
$Ƕ_m$ = sup {π (K, H): $degree$ of **A**, $degree$ of **E** ≤ **m**}. In short now Hilbert's problem converts to find a solution for $Ƕ_m$ in terms of **m** and possible arrangement of limit cycles.
After extreme efforts, we are not able to find $Ƕ_m$ for **m** = 2 till now. Bautin in 1952 provide $Ƕ_2 \geq 3$. $Ƕ_2 = 3$ is declared by Petroviskii , Shi and others gives examples of quadratic systems with minimum 4 limit cycles, gives $Ƕ_2 \geq 4$**.**$Ƕ_3 = 5$ when H and K are symmetric cubic polynomials [23]**.**For cubic systems, up to eight limit cycles can bifurcate **.**N.G. Lloyd showed $Ƕ_3 \geq 11$, for certain Hamiltonian systems by using bifurcation of limit cycle [23-25]
In 1923, Dulac claimed that polynomial systems donot limit cycles. But he cannot give its proof completely, new research tends to a proof of Dulac's theorem.
The system (1.2.1) can convert into a non-autonomous equation:
$d\Phi/d\sigma = \alpha(t)\Phi^3 + \beta(t)\Phi^2 - \lambda(\mathbf{m}\text{-}1)\,\sigma$ (1.3.8)
Transformation (1.6.1) facilitates the study of limit cycles in multidimensional system.
Recent findings in the study of limit cycles have led to a deeper understanding of the complex behavior of polynomial systems. The use of sophisticated mathematical techniques, such as non-convergent power series and bifurcation theory, has enabled researchers to tackle previously unsolved problems.
The transformation (1.6) maps the origin (Ŗ=0) to the origin (t = 0) and maps positive Ŗ-axis and positive σ- axis while preserving its shape and paths. It transforms the system into non autonomous equation
Existence of limit cycle can be confirmed by Poincaré-Bendixson theorem with no equilibrium points of bounded region. Bifurcation techniques also ensure existence of limit cycles when we vary the coefficients of system.

Formation of Periodic solutions

Introduction

This section commences with an exploration of a non-autonomous differential equation of first order of specific type

$$\frac{dz}{d\mathfrak{t}} = \alpha(\mathfrak{t})z^3 + \beta(\mathfrak{t})z^2 + \gamma(\mathfrak{t}) \quad (2.1.1)$$

z is complex and $\mathfrak{t}$ taken as real where $\mathfrak{Q}$, $\mathfrak{b}$ and γ are coefficients of real valued function. We fix the value $\omega \in R$ to fulfil the periodicity requirements and investigate the resulting number of solutions with boundary condition.

$$z(0) = z(\omega) \quad (2.1.2)$$

Although the coefficients of the equation are periodic, the solutions may or may not exhibit periodic behavior. Our focus is on the scenario where $\omega = 1$ and $\mathfrak{Q}$, $\mathfrak{b}$ are time-dependent polynomials. The equation (2.1.1) admits multiple periodic solutions, similar in nature to those of the real equation.

$$\frac{d\mathfrak{S}}{d\Theta} = \mathfrak{Q}(\Theta)\mathfrak{S}^3 + \mathfrak{b}(\Theta)\mathfrak{S}^2 - \zeta(\mathfrak{n} - 1)\mathfrak{S}$$

Our focus is on understanding the mechanisms governing the solution bifurcation, with the goal of establishing an upper bound for periodic creation and destruction. A solution is observed to undergo bifurcation, yielding up to $\boldsymbol{\mu}$ periodic solutions, each possessing multiplicity $\boldsymbol{\mu}$. The computational methodology for determining the origin 's multiplicity is described in [15].

Multiplicity analysis of origin

The multiplicity of $z = 0$ as solution of

$$\frac{dz}{d\mathfrak{t}} = \alpha(\mathfrak{t})z^3 + \beta(\mathfrak{t})z^2 + \gamma(\mathfrak{t}) \quad (2.2.1)$$

Will be zero of given below displacement function.

$$\mathfrak{q}_{\mathfrak{p}}; \mathcal{C} \longrightarrow z_{\mathfrak{p}}(\omega; 0, \mathcal{C}) - \mathcal{C}$$

The function described above in an open neighboring region of origin is holomorphic. Now we originate the way to find the multiplicity of $z = 0$ for $0 \leq \mathfrak{t} \leq \boldsymbol{\omega}$ and $\mathcal{C}$ lies in neighboring of 0, now express $z_{\mathfrak{p}}(\mathfrak{t}, 0, \mathcal{C})$ in power series.

$$z_{\mathfrak{p}} \text{in power} a_{\eta}(\mathfrak{t})\mathcal{C}^{\eta}$$

where

$$a_1(0) = 1 \text{ and } a_{\eta}(0) = 0 \text{ if } \eta > 1$$

Thus

$$\mathfrak{q}_{\mathfrak{p}}; \mathcal{C} \longrightarrow z_{\mathfrak{p}}(v; 0, \mathcal{C}) - \mathcal{C}$$

$$z_{\mathfrak{p}}(\mathfrak{t}, 0, \mathcal{C}) = \sum_{n=1}^{\infty} a_{\eta}(\mathfrak{t})\mathcal{C}^{\eta}$$

$$\mathfrak{q}_{\mathfrak{p}}(\mathcal{C}) = (a_1(\omega) - 1)\mathcal{C} + \sum_{\eta=1}^{\infty} a_{\eta}(\mathfrak{t})\mathcal{C}^{\eta} \qquad (2.2.2)$$

above equation inform us that the multiplicity of $z = 0$ is $\mu > 1$ if

$a_1(v) = 1, a_2(v) = a_3(v) = a_4(v) = 0 \cdots\cdots = a_{\mu-1}(v) = 0$ and $a_{\mu}(v) \neq 0$

Condition on Origin to become a center is

$a_1(\omega) = 1$ and $a_{\kappa}(\omega) = 0$ for all $\kappa > 1$

To using above two equations, set of differential equations that are linear for $a_n(\mathfrak{t})\, with$ initial conditions are $a_1(v) = 1$ and, $a_{\kappa}(v) = 0$, for $\kappa > 1$

It is noted that

$$\dot{a}_1(\mathfrak{t}) = \mathrm{a}_1(\mathfrak{t})\gamma(\mathfrak{t})$$

where

$$a_1(\mathfrak{t}) = exp\left(\int_0^{\omega} \gamma(\varsigma)d\varsigma\right)$$

Hence $\mu > 1$ iff

$$\int_0^{\omega} \gamma(\varsigma)d\varsigma = 0 \qquad (2.2.3)$$

Our target in the situation when $z = 0$ become a multiple alternative solution, suppose that (2.2.3) will be in this state into the transformation

$$\mathfrak{s} = z\, exp\left[-\int_0^{\omega} \gamma(\varsigma)d\varsigma\right] \qquad (2.2.4)$$

$$\dot{\mathfrak{s}} = \mathring{A}(\mathfrak{t})\mathfrak{s}^3 + \widehat{\mathfrak{B}}(\mathfrak{t})\mathfrak{s}^2 \qquad (2.2.5)$$

where

$$\mathring{A}(\mathfrak{t}) = \alpha(\mathfrak{t})exp\left(2\int_0^{1} \gamma(\varsigma)d\varsigma\right)$$

and

$$\beta(\mathfrak{t}) = \beta(\mathfrak{t})exp\left(\int_0^{1} \gamma(\varsigma)d\varsigma\right)$$

The functions α, β and $\boldsymbol{\gamma}$ that are given are periodic, then Å and β in the above equation are also periodic.

If multiplicity of $z = 0$ is greater than 1 [15]. Then the multiplicity of $\mathfrak{s} = 0$ is as a periodic solution of (2.2.5), when $\gamma(\mathfrak{t}) = 0$, we have an equation

$$\frac{dz}{d\mathfrak{t}} = \alpha(\mathfrak{t})z^3 + \beta(\mathfrak{t})z^2 \qquad (2.2.6)$$

For the above equation, $a_1(\mathfrak{t}) = 1$ and, for $\eta > 1$, the functions $a_{\eta}(\mathfrak{t})$ are given below

$$\dot{a}_{\eta} = \alpha \sum_{\substack{s+t+u=m \\ s,t,u \geq 1}} a_s a_t a_u + \beta \sum_{\substack{s+t=m \\ s,t \geq 1}} a_s a_t \qquad (2.2.7)$$

Recursive solution led to computational complexity and tedious integration by parts as n increases. So, limit our focus to $\eta \leq 10$.
Let

$$\eta_1 = a_1(\omega)$$

The focal values $\eta_t' s$, $t = 1,2,3,\ldots,8,9$ .... Is defined as the multiplicity of $z = 0$ that we express as μ is k if
$\eta_1 = 1$ and $\eta_2 = \eta_3 = \ldots\ldots\ldots\ldots = \eta_{k-1} = 0$ but $\eta_k \neq 0$
$\eth_k(t)$, $k \leq 9$ are generated by formulas given in literature and for k ≤ 10 is developed by Dr saima Akram. Now we describe some important formulas to calculate the focal values.

Evaluation of $a_j(t), \eta_j(t)$ and perturbation methodology

The theorem given below gives $a_j(t)$ and $\eta_j(t)$ for $j \leq 10$ [15,21,2,26-28]. These functions are important to find highest multiplicity of solution. But these calculations are complicated and to avoid mistakes we use programming language maple to evaluate these periodic solutions To indicate an infinite integral that is a bar over a function

$$\underline{a(t)} = \int_0^{\omega} \alpha(\varsigma)d\varsigma$$

Theorem (2.3.1)

For equation

$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2,$

functions $a_2, a_3, \ldots\ldots, a_{10}$ are given by the following formula.

$$a_2 = \bar{\beta}$$
$$a_3 = \beta^{-2} + \bar{\alpha}$$
$$a_4 = \beta + 2\bar{\beta}\bar{\alpha} + \overline{\beta^{-}\alpha}$$
$$a_5 = \bar{\beta}^4 + 3\bar{\beta}^2 + \bar{\beta}^2 + 2\bar{\beta}\bar{\beta}\alpha + \frac{3}{4}\alpha^{-2}$$
$$a_6 = \bar{\beta}^5 + 4\bar{\beta}^3\bar{\alpha} + \bar{\beta}^2\beta\alpha + 2\bar{\beta}\bar{\beta}^2\alpha + \frac{9}{2}\bar{\beta}\bar{\alpha} + 3\bar{\beta}\alpha\alpha - \frac{1}{2}\overline{\beta}\,\bar{\alpha}^2$$
$$a_7 = \bar{\beta}^6 + 5\bar{\beta}^4\alpha + \bar{\beta}^4\alpha + 4\bar{6}^3\beta\alpha + 2\bar{\beta}^3\alpha\bar{\beta} + \frac{17}{2}\bar{\beta}^2\bar{\alpha}^2 + 3\bar{\beta}^2\alpha\bar{\alpha} + \overline{(\bar{\beta}\alpha)}^2 + 2\overline{\bar{\beta}^2\bar{\alpha}}$$
$$+8\overline{\beta\alpha}\bar{\beta}\bar{\alpha}\overline{\beta\beta\bar{\alpha}} + \frac{5}{2}\alpha^3$$
$$a_8 = \bar{\beta}^7 + 6\beta^5\bar{\alpha} + \overline{\beta^5\alpha} + \bar{\beta}\beta\alpha + 2\overline{\beta^4}\alpha\bar{\beta} + 4\bar{\beta}^3\overline{\beta^2}\alpha + 3\overline{\beta^3}\alpha\bar{\alpha} + 3\overline{\beta^3}\alpha\bar{\beta}^2 + 3\bar{6}\alpha^3\bar{\alpha}\alpha$$
$$+\frac{27}{2}\bar{\beta}^3\bar{\alpha} - \frac{3}{2}\bar{\beta}^2\overline{\beta\alpha^{-2}} + 15\bar{\beta}^2\overline{\beta\alpha}\bar{\alpha} + 4\overline{6^2}\overline{\alpha}\alpha\bar{\beta} + \overline{\bar{\beta}^2\alpha\bar{\beta}\alpha} + 12\overline{\beta^2}\alpha\bar{\beta}$$
$$+8\bar{\beta}^2\alpha\overline{\beta\bar{\alpha}^2} + \bar{6}\overline{(\beta\bar{\alpha})^2} - \frac{1}{2}\overline{6\bar{\alpha}^2\bar{\alpha}} + 10\bar{\beta}\bar{\alpha}^3$$
$$a_9 = \overline{\beta^8} + 7\overline{\beta^6}\bar{\alpha} + 6\overline{\beta^5}\overline{\beta\alpha} + 2\overline{\beta^2\alpha\beta} + 5\bar{\beta}\overline{\beta^2\alpha} + 3\overline{\beta^4\alpha\bar{\alpha}} + 3\overline{\beta^3\alpha}\bar{\beta}^2 + 5\overline{\beta^4\bar{\alpha}}\alpha + \frac{39}{2}\bar{\beta}^4\bar{\alpha}^2$$
$$-2\bar{\beta}^3\overline{\beta\alpha^2} + 24\bar{\beta}^3\overline{\beta\alpha}\bar{\alpha} + 6\overline{\beta^3\alpha\bar{\alpha}}\bar{\beta} - 10\overline{\beta^3\alpha\bar{\beta}\bar{\alpha}} + 12\overline{\bar{\beta}\bar{\alpha}\beta^3\alpha} + +4\overline{\beta\bar{\alpha}\overline{\beta^3\alpha}}$$
$$+4\bar{6}^3\overline{\beta^3\alpha} + \frac{43}{6}\bar{\alpha}^3\bar{\beta}^2 + 4\overline{\beta\bar{\beta}\bar{\alpha}^3} + 4\bar{\beta}^2\alpha + 10\overline{\bar{\beta}\bar{\beta}\bar{\alpha}\beta^2\alpha} + \frac{15}{2}\alpha^{-2}\overline{\beta^2\alpha}$$
$$+2\overline{\bar{\beta}^2\beta^2\alpha} - 2\bar{\beta}^4\bar{\alpha} + 8\overline{\beta^3\bar{\alpha}\beta} + 2\overline{\bar{\beta}\bar{\beta}^2\alpha\beta\alpha} + 26\overline{\beta\alpha\bar{\beta}^2\alpha\beta} + 6\overline{\beta^2\alpha\alpha\bar{\alpha}}$$

$$-6\overline{\bar{\beta}^2\alpha\alpha\bar{\alpha}} + 12\overline{\bar{\beta}\bar{\beta}\bar{\alpha}\alpha} + 16\overline{\beta^2\bar{\alpha}\beta\bar{\alpha}\bar{\beta}} - 16\overline{\beta^3\alpha\bar{\beta}\alpha} + 9\bar{\beta}^2\overline{(\bar{\beta}\alpha)}^2 + 9\overline{(\bar{\beta}\alpha)}^2$$
$$-\overline{\bar{\beta}\alpha^3\bar{\beta}} + \frac{35}{8}\bar{\alpha}^4 - 6\overline{\bar{\alpha}\bar{\beta}}\beta\alpha^2 + 6\beta\overline{\bar{\alpha}\beta\overline{\alpha^2}} + 6\overline{\beta\bar{\alpha}\bar{\beta}\alpha^2} + 33\bar{\alpha}^2\bar{\beta}\overline{(\beta\alpha)} - 24\alpha^2\overline{\bar{\beta}\bar{\beta}}$$
$$+6\overline{\bar{\beta}^2\bar{\alpha}\alpha\bar{\alpha}} - 4\beta\overline{\alpha\bar{\beta}\overline{\alpha^2}}$$

Theorem (2.3.2):

For $z = 0$ as solution of equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \qquad (2.3.2)$$

has multiplicity b and $2 \le b \le 9$ iff $\eta_1 = 1$, $2 \le i \le b-1$ and $\eta_b \ne 0$ with

$$\eta_2 = \int_{\omega}^{0} \beta$$
$$\eta_3 = \int_{\omega}^{0} \alpha$$
$$\eta_4 = \int_{\omega}^{0} \alpha\bar{\beta}^2$$
$$\eta_6 = \int_{\omega}^{0} (\alpha\beta)$$
$$\eta_7 = \int_{\omega}^{0} (\alpha\bar{\beta}^4 + 2\alpha\bar{\alpha}\bar{\beta}^2)$$
$$\eta_8 = \int_{\omega}^{0} \alpha\bar{\beta}^5 + 3\alpha\bar{\alpha}\bar{\beta}^3 + \bar{\beta}^2\overline{\beta\alpha} - \frac{1}{2}\bar{\alpha}^3\beta$$
and
$$\eta_9 = \int_{\omega}^{0} \alpha\bar{\beta}^6 - 5\alpha\bar{\alpha}^4 - 2\bar{\beta}^3\overline{(\beta\bar{\alpha})} + 20\overline{\bar{\beta}\bar{\alpha}} + 2\bar{\bar{\alpha}}\beta\bar{\alpha}^2$$

$$\eta_{10} = \int_0^{\omega} \left(\alpha\underline{\beta}^7 - \frac{1235}{6}\alpha\underline{\alpha}\bar{\beta}^5 - \frac{970}{3}\alpha\underline{\alpha}^2\bar{\beta}^3 - 237\alpha\underline{\alpha}^2\bar{\beta}^3 - 24\alpha\underline{\alpha}^2\beta\bar{\beta}^2 - 70\underline{\alpha}\bar{\beta}^3\alpha^2 - 21\underline{\alpha}^4\beta \right.$$
$$\left. - 74\alpha\underline{\alpha}^3\bar{\beta} + \frac{5}{2}\underline{\alpha}^2\beta\bar{\beta}^4 + 32\bar{\beta}^4\alpha\overline{\beta\beta} - 16\beta\bar{\beta}^4\underline{\alpha} - 15\beta^5\alpha^2 - 366\beta\underline{\alpha}^2\overline{\beta\bar{\alpha}} - 8\beta\underline{\beta}^4\alpha\underline{\alpha}\right)$$

For every perturbation selected sequence which is obtained after perturbation of coefficients gives us minimum one periodic solution bifurcated from origin. We consider an equation of type (2.2.1) where value of multiplicity is fixed. We consider a neighborhood M throughout complex plane having one periodic solution for $z = 0$. Then result (2.4) in [13] tells us all possible number of periodic solutions are fixed after some minor change in coefficients where initial points are in M. We disturb coefficients, if necessary, then all

$b_2 = b_3 = \dots\dots\dots = b_{\upsilon-1} = 0$ but $b_\upsilon \ne 0$

Hence, we get a suggested periodic solution Ψ, that is non-trivial, it means we obtain same fixed no of periodic solutions after perturbation. Already known complicated solutions appear in coupled combination which ensure that Ψ is real. Then $\hat{W}_1$ be an adjacency of ψ and M_1 be an adjacency of $z = 0$ with condition

$$M_1 \cup A_1 \subset \text{M and } M_1 \cap A_1 = \emptyset$$

Initial points with each of M_1 and A_1 save number of periodic solutions under restricted perturbation of coefficients. We then again disturb the coefficients to get $b_2 = b_3 = \dots\dots\dots = b_{\upsilon-2} = 0$ but $b_{\upsilon-1} \ne 0$. In such situation multiplicity is υ-1. Now we attain second non-trivial real periodic solution, an initial point will be in M_1, while an initial point in A_1 ensure a real periodic solution exist. Therefore, multiplicity of zero solution is ʋ-2 and we attain 2 non-trivial periodic solution that are real. Following this concept and repeating these steps again and again, we get an equation of multiplicity ʋ-2, all solutions are distinct, nontrivial solutions and real.

Criteria for a center

To attain maximum multiplicity for different groups of coefficients of equation, we investigate the $\text{ƀ}_v = a_v(\omega)$ given in described above theorem until we attain a value v having property

$\text{ƀ}_v \neq 0$ for all if $\text{ƀ}_2 = \text{ƀ}_3 = \dots\dots\dots\dots = \text{ƀ}_{v-1} = 0$ then ɰ_{max} is maximum value of v.

We can compute ƀ_v by above discussed method when $z = 0$ is a center. Then we stop to find more ƀ_v .Results for $z = 0$ to be a centre, which are defined in [15], are repeated here because we need these statements to find ɰ_{max}.

Theorem

Let function ɣ which is differentiable with $\gamma(\omega) = \gamma(0)$, continuous and defined on $I = \gamma([0, \omega])$ that is

$$\alpha(t) = f(\gamma(t))\dot{\gamma},$$

$$\beta(t) = g(\gamma(t))\dot{\gamma}.$$

Then origin $z = 0$ is center of equation (2.3.2)

Corollary

Consider equation (2.3.2) where $\alpha(t)$ is a constant multiple of $\beta(t)$ and $\int_0^{\omega} \beta(t)\, dt = 0$ implies origin is a center.

Corollary

Let $\alpha(t)$ and $\beta(t)$ is identically zero and other has mean value zero. Then origin is a center as discussed in [16,17].

Corollary

Let $\alpha(t)$and $\beta(t)$ *having* odd powers of $sin(t)$ or cos(t),then origin is a center.

Summary of previous results

Here an equation of form

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \tag{2.5.1}$$

Above equations were initially used by Lins Neto in [13] where equation achive number of periodic solutions. He explained it with examples of coefficient $\theta(t)$ with degree one and $\mathfrak{Q}(t)$ has degree d_1, $\frac{d_1}{2} + 3$ periodic solutions exist. Alwash and Lloyd in [19] used polynomial in t or coefficients of function in $cos(t)$ and $sin(t)$ and explained the number of periodic solutions with examples that minimize the limitations given by Lins Neto [13]. Alwash consider class $\text{Ç}_{2,3}$,verify the result here.

Theorem

consider $\text{Ç}_{2,3}$ of equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

$\alpha(t), \beta(t)$ are of degree two and three respectively.

$$\alpha(t) = w + xt + yt^2$$

$$\beta(t) = r + st + ut^2 + vt^3$$

attain $\mu_{max}\ \text{Ç}_{2,3} = 8$.

The class $\text{Ç}_{l,1}$, l =1,2,3,..,6 ,$\mathfrak{Q}(t)$ and $\theta(t)$ has degree l and 01respectively is explored by Alwash and Lloyd in [15-17]. If n_l *represent* $\mu_{max}(C_{l,1})$ then $n_1 = 3, n_2 = 4, n_3 = 4, n_4 = 5$ $n_5 = 5$ and $n_6 \geq 7$ are results explored by them In [15-17] .Alwash gives Ң_l class of equations of type (2.3.2) with $\mathfrak{Q}(t)$ and $\theta(t)$ homogeneous polynomials in $cos(t)$ and $sin(t)$of degree 2l and l respectively. Hilbert number is calculated for these classes. Then they have result, if $\mu_{max}(\text{Ң}_l) = f_l$. Then $f_1 = 5, f_2 = 5$ and $f_3 \geq 7$.N. Yasmin in [20,21] let the classes $C_{1,l}, l = 1,2,3,4,5$ with $\mathfrak{Q}(t)$ and $\theta(t)$ of degree 1 and l respectively. Then she conclude that, if $n_l = \mu_{max}(C_{1,l})$ then $n_1 = 3, n_2 = n_3 = 4, n_4 = n_5 = 8$. S. Akram derive formula for maximum multiplicity 10 , also she verify and modify previous results by using new developed formula [26-29]. A short overview of results of different classes obtained till now are given in the following table. Most of results are improved by Dr S. Akram.

<table>
<tr><th>α/β</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th><th>11</th><th>12</th><th>14</th></tr>
<tr><td>1</td><td>/2/</td><td>/3/</td><td>/4/</td><td>/5/</td><td>/6/</td><td>(< 7 <)</td><td></td><td></td><td>‘’10’’</td><td></td><td>*8*</td><td></td><td>,10,</td></tr>
<tr><td>2</td><td>4*</td><td>“4”</td><td>8</td><td>(((8)))</td><td>{8}</td><td>(< 8 <)</td><td></td><td></td><td>‘’10’’</td><td></td><td>*7*</td><td></td><td>,8,</td></tr>
<tr><td>3</td><td>4*</td><td>“8”</td><td>(((5)))</td><td>‘’8’’</td><td>(< 8 <)</td><td></td><td>‘’10’’</td><td>‘’8’’</td><td>‘’10’’</td><td>\7\</td><td>*7*</td><td>{8}</td><td>,8,</td></tr>
<tr><td>4</td><td>5*</td><td>7</td><td>{8}</td><td>*7</td><td></td><td></td><td></td><td>‘’8’’</td><td>:8:</td><td>\7\</td><td>;8;</td><td></td><td></td></tr>
<tr><td>5</td><td>5*</td><td>{7}</td><td></td><td></td><td></td><td></td><td>‘’8’’</td><td>‘’8’’</td><td>:8:</td><td>‘’9’’</td><td>;8;</td><td></td><td></td></tr>
<tr><td>6</td><td>7*</td><td></td><td></td><td></td><td></td><td></td><td></td><td>‘’8’’</td><td>:7:</td><td></td><td>;8;</td><td>‘’8’’</td><td></td></tr>
<tr><td>7</td><td></td><td></td><td>{10}</td><td>{10}</td><td>{8}</td><td>{8}</td><td>‘’8’’</td><td>‘’8’’</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>8</td><td></td><td></td><td>‘’8’’</td><td></td><td>{8}</td><td>{8}</td><td>{8}</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>9</td><td>|||10|</td><td>|||10 |||</td><td>|||7|||</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>10</td><td></td><td></td><td>{10}</td><td>{10}</td><td>{8}</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>12</td><td></td><td></td><td></td><td></td><td></td><td>{8}</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Amar result given in (*..*), Zahid Saleem results given in (,..,),M.Irfan results given in (;..;), M.Iftikhar Hussain results given in(\..\), Arooj result given in(:..:),Entriwithin (..(..*) express results obtained by Alwash. Results given in (/../) are used of N.Yasmin in [20-21]. Results with bar are given by M.Ashraf. Results with in ((( ))) are deduced by Jamil Ahmad. Results in (*..) are calculated by Gul Hassan. Numbers given within (“”) show results given by Dr Saima Akram and Allah Nawaz. Results given in (< <) are of Azra Aziz and entries with (||| |||)resultt found by M.Irfan. My work is on classes $Ç_{18,1} > 10$, $Ç_{18,2} > 8$, $Ç_{18,3} > 10$.

Bifurcating periodic solutions

Initiation

Assume
$$\frac{dz}{dt} = \mathfrak{Q}(t)z^3 + \beta(t)z^2 \qquad (3.1.1)$$

with α and β both polynomial functions depending on t and z is complex, above equation considered by Lins Neto [13], in which pugh’s question is discussed. We concern with number of periodic solutions bifurcated from origin [15], now our concern is about equation (3.1.1) to investigate the optimum multiplicity with same origins related to theorem (2.3.2). These focal values attain by Maple.

Maple to compute focal values

To calculate periodic solutions of equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \qquad (3.2.1)$$

Focal values η_β are already discussed with $\alpha(t)$ and $\beta(t)$ *of degree* higher than 2 which cannot be estimated by hand calculations. Maple can help us in evaluation of focal values. With the help of Maple commands polynomials can be integrated, expanded and factorized have explained key commands for computing focal values. we locate η_2 for equation(3.2.1).

$$\alpha(t) = a_r + b_r t + c_r t^2 + d_r t^2 + e_r t^4 + f_r t^5 + g_r t^6 + h_r t^7 + i_r t^8$$
$$\beta(t) = k_p + l_p t + m_p t^2$$

In maple we write it as

Alpha:= a_p+b_p.t+c_p.t^2+d_p.t^3+e_p.t^4+f_p.t^5+g_p.t^6+h_p.t^7+i_p.t^8;

Beta:= k_p+l_p.t+m_p.t^2;

Formula for η_2 is

$$\eta_2 = \int_0^1 \beta(t)\, dt.$$

We calculate integral in maple.

Maple integration

The "int" function is performed with respect to a variable to estimate the definite or indefinite integral of polynomial. For indefinite integration, there is a name in second argument t. If form of second argument is $t = x_0..y_0$ where x_0 *and* y_0 are initial and final points of interval of integration. In this topic some time we use a_r and sometimes a_p both have same meanings. We write η_2 in Maple as

$$eta2 := int(beta, t = 0..1);$$

then we get

$$\eta_2 = k_p + l_p/2 + m_p/3$$

To evaluate $\eta_3 = \int_0^1 \alpha(t)dt$ .We write it in Maple.

$$eta3 := int(alpha, t = 0..1);$$

And obtain

$$\eta_3 = a_r + \frac{b_r}{2} + \frac{c_r}{3} + \frac{d_r}{4} + \frac{e_r}{5} + \frac{f_r}{6} + \frac{g_r}{7} + \frac{h_r}{8} + \frac{i_r}{9}$$

theorem(2.3.2) inform us

$$\eta_4 = \int_0^1 \alpha(t).\bar{\beta}(t)dt.$$

where

$$\bar{\beta}(t) = \int \beta(t)dt.$$

similarly $beta1 = betabar = int(beta, t)$

So $\beta_1 = k_p t + l_p t^2/2 + m_p t^3/3$

For attaining η_4 we calculate value $\alpha(t)\bar{\beta}(t)$.

Extend an expression in Maple.

Expansion program is to give out products of polynomial. Performed as

$$gemma := \alpha(t)\bar{\beta}(t);$$

then

$$gemma := expand(alpha * beta1);$$

$$gemma := (a_r + b_r t + c_r t^2 + d_r t^2 + e_r t^4 + f_r t^5 + g_r t^6 + h_r t^7 + i_r t^8)(k_p t + l_p t^2/2 + m_p t^3/3)$$

$gemma4$ is calculating using syntax for η_2 and η_3

$$gemma4 := \frac{2}{35}m_p l_p + \frac{1}{15}i_p l_p + \frac{1}{17}m_p k_p + \frac{1}{13}h_p l_p + \frac{1}{19}i_p k_p + \frac{1}{18}g_p l_p + \frac{1}{7}h_p k_p + \frac{1}{12}l_p f_p + \frac{1}{6}g_p k_p + \frac{1}{13}k_p l_p + \frac{1}{9}f_p k_p + \frac{1}{11}d_p l_p + \frac{1}{5}e_p k_p + \frac{1}{8}l_p c_p + \frac{1}{6}k_p d_p + \frac{1}{7}b_p l_p + \frac{1}{5}c_p k_p + \frac{1}{7}a_p l_p + \frac{1}{2}b_p k_p + \frac{1}{3}a_p k_p.$$

To attain multiplicity greater than 3 , put $\eta_2 = \eta_3 = 0$

We get values of a_p *and* k_p after putting $\eta_2 = 0$ and $\eta_3 = 0$

$$a_p = -\frac{b_p}{2} - \frac{c_p}{3} - \frac{d_p}{4} - \frac{e_p}{5} - \frac{f_p}{6} - \frac{g_p}{7} - \frac{h_p}{8} - \frac{i_p}{9}$$

and

$$k_p = -l_p/2 - m_p/3$$

In Maple, substitution command is used to putting above values in η_4 .

Substitution of Value in Expression.

We follow format syntax when substitute value of coefficient to change some another value.

$eta4 := expand(subs\left\{a_p = -\frac{b_p}{2} - \frac{c_p}{3} - \frac{d_p}{4} - \frac{e_p}{5} - \frac{f_p}{6} - \frac{g_p}{7} - \frac{h_p}{8} - \frac{i_p}{9},\ k_p = -\frac{l_p}{2} - \frac{m_p}{3}\right\}, \{gemma4\}));$

We obtain η_4

$$\eta_4 := \left\{\frac{1}{120}m_p l_p + \frac{7}{1375}h_p l_p + \frac{5}{1470}g_p l_p + \frac{7}{1628}f_p l_p + \frac{5}{1658}e_p l_p + \frac{1}{230}d_p l_p + \frac{1}{210}c_p l_p + \frac{1}{310}i_p l_p\right\}.$$

We can factorize an expression in maple.

Factorization in Maple.

We factorize multi-variable polynomial of algebraic number coefficients.

"factor" function is used for this purpose.

Suppose $eta\ 41 := \frac{1}{120} * m_p * l_p + \frac{7}{1375} * m_p * h_p * l_p + \frac{5}{1470} * m_p * g_p * l_p + \frac{7}{1628} * m_p * f_p * l_p;$

The factorization of eta 41 is

$$eta41 := factor(eta41);$$

It gives

$\eta_4 = m_p l_p \left(\frac{1}{120} + \frac{7}{1375}h_p + \frac{5}{1470}g_p + \frac{7}{1628}f_p\right)$.All above discussed commands help us to find focal values.

Periodic solution of class $C_{18,1}$

Consider $C_{18,1}$,equation of form

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

$\alpha(t)$ has degree 18,$\beta(t)$ has degree 1 respectively that is

$\alpha(t) = c + dt + et^2 + ft^3 + gt^4 + ht^5 + it^6 + jt^7 + kt^8 + lt^9 + mt^{10} + nt^{11} + ot^{12} + pt^{13} + qt^{14} + rt^{15} + st^{16} + ut^{17} + vt^{18}$

and

$$\beta(t) = a + bt$$

Then by previous theorem

$\eta_2 = a + \frac{b}{2}$

also

$\eta_3 = c + \frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}$

Origin multiplicity is $\mu = 2$ if η_2 is not equal to zero and $\mu = 3$ if we have $\eta_2 = 0$ and $\eta_3 \neq 0$, for origin multiplicity higher then 3 consider $\eta_2 = 0 = \eta_3$ implies

$$a = -\frac{b}{2} \qquad (3.3.1)$$

$$c = -(\frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}) \qquad (3.3.2)$$

Now

$$\alpha(ț) = -(\frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}) + dț + eț^2 + fț^3 + gț^4 + hț^5 + ið^6 + jț^7 + kț^8 + lț^9 + mț^{10} + nț^{11} + oț^{12} + pț^{13} + qț^{14} + rț^{15} + sț^{16} + uț^{17} + vț^{18} \qquad (3.3.3)$$

$$\beta(ț) = a + bț \qquad (3.3.4)$$

$$\eta_4 = \frac{1}{2793510720}\{b(7759752e + 11639628f + 13302432g + 13856700h + 13579566j + 13168064k + 12697776l + 12209400m + 11724900n + 11255904o + 10808226p + 10384374q + 9984975r + 9609600s + 9257248u + 8926632v)\}$$

As for $\eta_4 = 0$,b= 0 or $\left\{e = \frac{1}{759752}\{11639628f + 13302432g + 13856700h + 13579566j + 13168064k + 12697776l + 12209400m + 11724900n + 11255904o + 10808226p + 10384374q + 9984975r + 9609600s + 9257248u + 8926632v\right\}$

As b is not equal to zero,if we chose b=0 then $\beta(ț)$ becomes equal to zero and in this case center becomes origin so we chose $\left\{e = \frac{1}{759752}\{11639628f + 13302432g + 13856700h + 13579566j + 13168064k + 12697776l + 12209400m + 11724900n + 11255904o + 10808226p + 10384374q + 9984975r + 9609600s + 9257248u + 8926632v\right\}$

And after multiple steps done in maple we evaluate eta5 by using formula discussed earlier

η_5=-1/642507465600(b^2)(254963280e+382444920f+433437576g+446185740h+440391120i+425904570j+407703520k+388328688l+369072720m+350574510n+333125100o+316829370p+301693392q+287672385r+274697280s+262689440u+251568720v)

Expression become complicated and further calculation to find focal values are not possible. For simplicity choose some coefficients zero.

Theorem

Consider class $C_{18,1}$ of type

$$\frac{dz}{dț} = \alpha(ț)z^3 + \beta(ț)z^2$$

with $\alpha(ț), \beta(ț)$ of degree 18 and 1 respectively.

$$\alpha(ț) = a + dț^3 + bț + eț^4 + fț^5 + \boldsymbol{s}ț^{\mathbf{18}}$$

and

$$\beta(ț) = u + yț$$

Then $\mu_{max}\ C_{18,1} \geq 10$.

Proof

We are interested to compute $ƀ_p$, $p = 2,3,..,10$ for multiplicity of origin. For above coefficients we'll obtain

$$ƀ_2 = u + \frac{y}{2}$$

$$ƀ_3 = a + \frac{b}{2} + \frac{d}{4} + \frac{e}{5} + \frac{f}{6} + \frac{s}{19}$$

by using previous discussed results we can say that multiplicity of the origin $z = 0$ is 2 if $\text{ƀ}_2 \neq 0$ and μ *will be equal to* 3 if $\text{ƀ}_2 = 0$ *and* $\text{ƀ}_3 \neq 0$.For attaining multiplicity of order more than 3, take $\text{ƀ}_2 = \text{ƀ}_3 = 0$ then

$$u = -\frac{y}{4} \tag{3.3.1}$$

$$a = -\frac{b}{2} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{s}{19} \tag{3.3.2}$$

Above two equations are utilized to get new form of coefficients $\alpha(\text{ƫ})$ *and* $\beta(\text{ƫ})$ to measure η_4 .Therefore

$$\alpha(\text{ƫ}) = -\frac{b}{2} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{s}{19} + d\text{ƫ}^3 + b\text{ƫ} + e\text{ƫ}^4 + f\text{ƫ}^5 + \boldsymbol{s}\text{ƫ}^{\mathbf{18}} \tag{3.3.3}$$

$$\beta(\text{ƫ}) = -\frac{y}{2} + \frac{y}{2}\text{ƫ} \tag{3.3.4}$$

$$\text{ƀ}_4 = \frac{1}{95760} y(306s + 475f + 456e + 399d)$$

Now as we are interested in higher multiplicity, so we put $\text{ƀ}_4 = 0$ i.e.

$$y(306s + 475f + 456e + 399d) = 0$$

Implies that either $y = 0$

or $$\mathrm{d} = \left(-\frac{306}{399}s - \frac{475}{399}f - \frac{456}{399}e\right) \tag{3.3.5}$$

If we choose $y = 0$ then we get $\beta(\text{ƫ}) = 0$ and also $\text{ƀ}_2 = 0$ then corollary$(2.4.3)$ gives average values of alpha zero and origin become center, So we consider $y \neq 0$. Thus we have other possibility

$$\mathrm{d} = \left(-\frac{306}{399}s - \frac{475}{399}f - \frac{456}{399}e\right)$$

Thus $\alpha(\text{ƫ})$ and $\beta(\text{ƫ})$ becomes

$$\beta(\text{ƫ}) = -\frac{y}{4} + \frac{y}{4}\text{ƫ}$$

and $\text{ƀ}_5 = y^2\left(\frac{110160s + 24035\,f + 9614e}{1695909600}\right)$

If we want $\text{ƀ}_5 = 0$.

Then value of f will be

$$f = -\frac{110160}{24035}s - \frac{9614}{24035}e \tag{3.3.6}$$

as $y \neq 0$ (proved).

To measure η_6 use above value and we get

$$\text{ƀ}_6 = \frac{17sy(329015115y^2 \pm 586753470s + 26025098e)}{7494235424676000}$$

Furthermore if we assume $\text{ƀ}_6 = 0$,it is possible only when

either $y = 0$ or

$$s = \frac{329015115y^2 + 26025098e}{586753470} \tag{3.3.7}$$

Since we assumed $y \neq 0$ (proved) and we choose value of

$$s = \frac{329015115y^2 + 26025098e}{586753470}$$

We utilize $(3.5.13)$ to measure ƀ_7

$$\beta(\text{ƫ}) = -\frac{y}{4} + \frac{y}{4}\text{ƫ}$$

$$\text{ƀ}_7 = \frac{17\,y^2(68445y^2 + 5414\,e)(32804491597665y^2 + 35417943045810b + 2271836331643601e)}{7395294608016249114180000`}$$

If we assume $\text{ƀ}_7 = 0$ have possibility either $y = 0$ gives us origin become center ,so we choose $y \neq 0$,and put

$e = \frac{329015115}{5414}y^2$ so

$\text{ƀ}_8 = -\frac{1}{4696732469583296965168856234528420728000000}(y(68445y^2 + 5414e)(862262734656893672418367739933476895y^4 + 1367341194711520884710556059609573 98y^2 e + 54388068157683888308183 59e^2)$

$$\text{ƀ}_9 = \frac{-1}{843107360210850518 7840}y^5(14070900605901087y + 110438634779438 7280)$$

$\text{ƀ}_{10} = \frac{-1936999299478882439386469621301498000907446551992778251463896038960388829158240028651524039957742244518028682179512850095910706}{270712938877709517533878510555954978046704256804867599527366167601986060695571625491436430767820328883064196511105555322315227882252778863534215861969315635054199}$

Thus $\mu_{max}(C_{18,1}) = 10$ if all above eq

Theorem

Consider the equation of the type

$$\frac{dz}{dt} = \mathfrak{Q}(t)z^3 + \beta(t)z^2 \qquad (3.3.8)$$

where

$$\alpha(t) = \left(\frac{212050812612270013440}{42330274406680} - \frac{1}{2}\varepsilon_1 - \frac{205}{678}\varepsilon_2 - \frac{947}{57632}\varepsilon_3\right) + \left(\frac{639071545728297408}{88561545625} - \left(\frac{223}{125} + \varepsilon_4\right)^2\right)t + \left(\frac{164219479054969790021}{1794690397955140625} - \frac{1829}{95936}\varepsilon_3 + \left(\frac{651825}{72156} + \varepsilon_5\right)^2\right)t^3 + \left(-\frac{508874480008576}{518569854079055625} + \frac{1110400}{482969}\varepsilon_3 + \left(\frac{1700981}{30869} + \varepsilon_6\right)^2\right)t^4 + \left(\frac{25857193916770150909}{57095876105491275} - \frac{207}{19}\varepsilon_3 - \frac{98}{119}\varepsilon_2\right)t^5 + \left(-\frac{25419174551092002299040}{115264633790709} - \frac{7}{72}\varepsilon_4 - \frac{11}{129}\varepsilon_7 - \frac{11}{198}\varepsilon_2 +\right)t^{18}$$

$$\beta(t) = \left(-\frac{45464}{786025} - \frac{1}{2}\varepsilon_1\right) + \varepsilon_8 + \left(\frac{3452732}{456025} + \varepsilon_1\right)t$$

where ε_p, $1 \le p \le 8$, are non-zero chosen values and every ε_p is smaller than ε_{p-1} then (2.3.2) has eight different real periodic solution that are non-trivial.

Proof

Multiplicity of origin is 10 due to selected coefficients $\varepsilon_p = 0$ for $1 \le p \le 8$. Select $\varepsilon_1 \ne 0$ but all other $\varepsilon_p = 0$,$2 \le p \le 8$ then it noted that,

$\varepsilon_2 = \varepsilon_3 = \ ... = \varepsilon_7 = 0$ but $\varepsilon_8 \ne 0$ and ƀ_9 is constant multiple of ε_1, so $\mu = 9$.

In this procedure multiplicity is reduced by 1. Take $\varepsilon_2 \ne 0$ but $\varepsilon_3 = \varepsilon_4 = \ ... = \varepsilon_8 = 0$; we have eta2 to eta5 all zero but then $\eta_6 \ne 0$ and η_7 is constant multiple of ε_2, so $\mu = 8$. Here also multiplicity is decrease by one.

If the value of ε_2 is small enough then we have two real periodic solution that are non-trivial. Continuing in this way, we have eight real periodic solutions that are non-trivial.

Corollary

Take $\alpha(t)$ and $\beta(t)$ given in result (3.3.8), then

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 + \gamma(t)z + \delta. \qquad (3.3.9)$$

have ten periodic solutions provided that γ and δ are small enough.

Proof

If $\gamma(t) = 0$, $\mathfrak{Q} = 0$ and $\mu = 2$ then equation (3.5.15) gives eight real periodic solutions. If $\gamma(t)$ is not equal to zero, then $\mu = 1$ and then same arguments used in above theorem we have nine real periodic solutions. Since $z = 0$ is another solution therefore we have eight real periodic solutions.

Periodic solution of class $C_{18,2}$

Let's $C_{18,3}$ express equation of the type

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

where $\alpha(t)$ has degree 18 and $\beta(t)$ has degree 2 respectively that is,

$$\alpha(t) = c + dt + et^2 + ft^3 + gt^4 + ht^5 + it^6 + jt^7 + kt^8 + lt^9 + mt^{10} + nt^{11} + ot^{12} + pt^{13} + qt^{14} + rt^{15} + st^{16} + ut^{17} + vt^{18}$$

and

$$\beta(t) = a + bt + wt^2$$

Then by using formula of eta2 given in literature

$$\eta_2 = a + \frac{b}{2} + \frac{w}{3}$$

and

$$\eta_3 = c + \frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}$$

Origin multiplicity will be $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_2 = 0$ and $\eta_3 \neq 0$, for origin multiplicity higher then 3 we consider $\eta_2 = \eta_3 = 0$,it gives us

$$u = -\left(\frac{b}{2} + \frac{w}{3}\right) \qquad (3.4.1)$$

$$c = -\left(\frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}\right)$$

(3.3.2)

Now function $\alpha(t)$ and $\beta(t)$ becomes

$$\alpha(t) = -\left(\frac{d}{2} + \frac{e}{3} + \frac{f}{4} + \frac{g}{5} + \frac{h}{6} + \frac{i}{7} + \frac{j}{8} + \frac{k}{9} + \frac{l}{10} + \frac{m}{11} + \frac{n}{12} + \frac{0}{13} + \frac{p}{14} + \frac{q}{15} + \frac{r}{16} + \frac{s}{17} + \frac{u}{18} + \frac{v}{19}\right) + dt + et^2 + ft^3 + gt^4 + ht^5 + it^6 + jt^7 + kt^8 + lt^9 + mt^{10} + nt^{11} + ot^{12} + pt^{13} + qt^{14} + rt^{15} + st^{16} + ut^{17} + vt^{18}$$

a (3.3.3)

$$\beta(t) = -\left(\frac{b}{2} + \frac{w}{3}\right) + bt + wt^2 \qquad (3.4.2)$$

$$\eta_4 := \left\{\frac{7}{1440}bj + \frac{7}{1485}kb + \frac{1}{220}lb + \frac{5}{1144}mb + \frac{55}{13104}nb + \frac{11}{2730}ob + \frac{13}{3360}pb + \frac{91}{24480}qb + \frac{35}{9792}rb + \frac{10}{2907}sb + \frac{17}{5130}ub + \frac{17}{5320}vb + \frac{1}{360}gw + \frac{5}{1512}hw + \frac{1}{280}iw + \frac{35}{9504}jw + \frac{1}{270}kw + \frac{21}{5720}lw + \frac{5}{1386}mw + \frac{11}{3120}nw + \frac{5}{1456}ow + \frac{143}{42840}pw + \frac{7}{2160}qw + \frac{65}{20672}rw + \frac{7}{2295}sw + \frac{85}{28728}uw + \frac{3}{1045}vw - \frac{1}{360}dw + \frac{1}{560fw} + \frac{1}{360}eb + \frac{1}{240}fb + \frac{1}{210}gb + \frac{5}{1008}hb + \frac{5}{1008}ib\right\}$$

Expression become complicated and further calculation to find focal values are not possible. For simplicity choose some coefficients of polynomial alpha and beta and assume the simplified class.

Theorem

Let $C_{18,2}$ is an equation of kind

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

with $\alpha(t)$ and $\beta(t)$ are of degree18 and 2respectively.

$$\alpha(t) = a + bt + ct^2 + et^4 + \boldsymbol{s}t^{\mathbf{18}}$$

and

$$\beta(t) = u + wt2$$

Then $\mu_{max}\ C_{18,2} \geq 8$.

Proof

We compute $\eta_{\mathfrak{b}}, \mathfrak{b} = 2,3,..,8$ to calculate multiplicity of origin.Thus for chosen polynomial coefficients of class, we'll use formulas and evaluate

$$\eta_2 = u + \frac{w}{3}$$

and

$$\eta_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{e}{5} + \frac{s}{19}$$

Now multiplicity of that origin $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_3 \neq 0$. To compute the greater multiplicity, we took $\eta_2 = 0$ and $\eta_3 = 0$ after putting this we obtain

$$u = -\frac{w}{3} \qquad (3.4.3)$$

and

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{e}{5} - \frac{s}{19} \qquad (3.4.4)$$

We use above two equations to evaluate η_4 also modified $\alpha(\mathfrak{t})$ and $\beta(\mathfrak{t})$ are

$$\alpha(\mathfrak{t}) = -\frac{b}{2} - \frac{c}{3} - \frac{e}{5} - \frac{s}{19} - \frac{s}{19} + +b\mathfrak{t} + c\mathfrak{t}^2 + e\mathfrak{t}^4 + \boldsymbol{s}\mathfrak{t}^{\mathbf{18}} \qquad (3.4.5)$$

$$\beta(\mathfrak{t}) = -\frac{w}{3} + \frac{w}{3}\mathfrak{t} \qquad (3.4.6)$$

$$\eta_4 = \frac{-w}{75240}(-216s - 209e + 209b)$$

Now for $\eta_4 = 0$ i.e.

$$w(-216s - 209e + 209b) = 0$$

So, either $w = 0$

or $\quad e = -\frac{216}{209}s + b \qquad (3.4.7)$

If $w = 0$ then we find $\beta(\mathfrak{t}) = 0$ and also $\eta_3 = 0$ .Using eta3 equal to zero we get average values of alpha also zero then by use of corollary given already, the origin becomes center if $w = 0$ hereafter we took $w \neq 0$. Thus we have

$$e = -\frac{216}{209}s + b$$

Thus $\alpha(\mathfrak{t})$ and $\beta(\mathfrak{t})$ becomes

$\alpha(\mathfrak{t}) =$

$\beta(\mathfrak{t}) = -\frac{y}{4} + \frac{y}{4}\mathfrak{t}$

using above equations we calculate

$\eta_5 = w^2(\frac{313344s - 7429b}{545113800})$

If $\eta_5 = 0$.

Then we take

$b = \frac{313344}{7429}(\frac{s}{11}) \qquad (3.4.8)$

As $w \neq 0$ (proved).

We use value of b and measure η_6 and for this $\mathfrak{Q}(\mathfrak{t})$ and $\mathfrak{G}(\mathfrak{t})$ becomes

$\beta(\mathfrak{t}) = -\frac{y}{4} + \frac{y}{4}\mathfrak{t}$

$$\eta_6 = \frac{(17sw)(9634447461797w^2 + 4508126032896c)}{99427333128985242501120} \qquad (3.4.9)$$

Since $s, w \neq 0$ (proved) and we took

$s = -\frac{4210495}{11854848}\, w^2$

Using value of s given in above equation to measure η_7 thus $\alpha(\mathfrak{t})$ and$\beta(\mathfrak{t})$ become

$\beta(\mathfrak{t}) = -\frac{y}{4} + \frac{y}{4}\mathfrak{t}$

$$\eta_7 = \frac{-1927\, w^4(-9634447461797w^2+4508126032896c)}{9942733312898524250112 0`}$$

If we put $\eta_7 = 0$ then y= 0 then the origin is center or

$$\eta_8 = = \frac{178750529205659953007929 0154297}{2571224385578699276765755284020369817600} w^7$$

Thus $\mu_{max}(C_{18,3}) = 8$ if all results attained in above equation hold and $(w)(s) \neq 0$.

Theorem

Consider equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \tag{3.4.10}$$

where

$$\alpha(t) = \left(-\frac{964707}{13149076}q^2 - \frac{1087}{153192}\varepsilon_2 - \frac{9}{160}\varepsilon_1\right) + \left(\frac{8158131596892117133}{2458625098192963360}q^2 + \varepsilon_1\right)t^2 + \left(\frac{1153909}{2114490}q^2 + \varepsilon_2\right)t^7 + \left(-\frac{1533993897}{140178209}q^2 - \frac{9505}{4408}\varepsilon_2 + \varepsilon_4\right)t^8 + \left(\frac{724238852}{13121066493}q^2 + \frac{81640}{80001}\varepsilon_2 + \varepsilon_5\right)t^{18}$$

$$\beta(t) = \left(\frac{55219897193640}{140709001087} - \frac{1}{2}\,\varepsilon_1\right) + \varepsilon_6 + \left(-\frac{1104386394387280}{1407060591087} + \varepsilon_1\right)t\text{^}2$$

If i, $1 \leq p \leq 6$, if we select all these are not equal to zero and also each ε_p is smaller as ε_{p-1} then above equation gives 8 real periodic solutions that are not trivial solutions.

Proof

The coefficients that are selected above gives origin multiplicity 8 if $\varepsilon_i = 0$ for $1 \leq i \leq 6$. Pick $\varepsilon_1 \neq 0$ but $\varepsilon_i = 0$ for $2 \leq i \leq 6$ gives us $\eta_2 = \eta_3 = \dots\dots\dots\dots = \eta_5 = 0$ but $\eta_7 \neq 0$ also η_7 is constant multiple of ε_i ,gives $\mu = 7$.

So one multiplicity is decrease. Then we take $\varepsilon_2 \neq 0, \varepsilon_3 = \varepsilon_4 = \dots\dots = \varepsilon_5 = 0$; we have $\eta_2 = \eta_3 = \eta_4 = 0$, $\eta_6 \neq 0$ and η_6 is constant multiple of ε_2, so $\mu = 6$. Here also one multiplicity is decreased. ε_2 is small and we attain two real periodic solutions that are non-trivial. Repeating in this way, we find eight real periodic solutions that are non-trivial.

Corollary

Taking $\alpha(t)$ and $\beta(t)$ used in above result, equation.

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 + \gamma(t)z + \delta \tag{3.4.11}$$

Exhibits ten real periodic solutions in the limit of small γ and δ .

Proof

If $\gamma(t) = 0, \delta = 0$ and $\mu = 2$ then above equation gives 8 real periodic solutions. If $\gamma(t)$ is not equal to zero, then $\mu = 1$ and then using concept and logic used in above result we attain nine real periodic solutions. As $z = 0$ also solution therefore ten real periodic solutions are evaluated.

Periodic solution of $C_{18,3}$

Elaborate $C_{18,3}$ of type

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

with $\alpha(t)$, $\beta(t)$ of degree 18 and 3 respectively

$$\alpha(t) = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11} + mt^{12} + nt^{13} + ot^{14} + pt^{15} + qt^{16} + rt^{17} + st^{18}$$

and

$$\beta(t) = u_i + v_i t + w_i t^2 + y_i t^3$$

Using formula of eta 2 in literature

$$\eta_2 = u_i + \frac{v_i}{2} + \frac{w_i}{3} + \frac{y_i}{4}$$

$\eta_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} + \frac{f}{6} + \frac{g}{7} + \frac{h}{8} + \frac{i}{9} + \frac{j}{10} + \frac{k}{11} + \frac{l}{12} + \frac{m}{13} + \frac{n}{14} + \frac{o}{15} + \frac{p}{16} + \frac{q}{17} + \frac{r}{18} + \frac{s}{19}$

multiplicity will be $\mu = 2$ only when eta2 is not equal to zero, $\mu = 3$ when $\eta_2 = 0$ and eta3 is not equal to zero, for maximum multiplicity we consider $\eta_2 = 0, \eta_3 = 0$,it gives us

$$u_i = -\left(\frac{v_i}{2} + \frac{w_i}{3} + \frac{y_i}{4}\right) \qquad (3.5.1)$$

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12} - \frac{m}{13} - \frac{n}{14} - \frac{o}{15} - \frac{p}{16} - \frac{q}{17} - \frac{r}{18} - \frac{s}{19}$$

(3.3.2)

Now

$$\alpha(t) = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12} - \frac{m}{13} - \frac{n}{14} - \frac{o}{15} - \frac{p}{16} - \frac{q}{17} - \frac{r}{18} - \frac{s}{19} + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11} + mt^{12} + nt^{13} + ot^{14} + pt^{15} + qt^{16} + rt^{17} + st^{18} \qquad (3.5.2)$$

$$\beta(t) = -\frac{v}{2} - \frac{w}{3} - \frac{y}{4} + vt + wt^2 + yt^3 \qquad (3.5.3)$$

$$\eta_4 := \left\{-\frac{51}{320}bo + \frac{21}{280}lcn - \frac{13}{250}bps + \frac{19}{340}dnr - \frac{1}{510}bq - \frac{1}{620}cp + \frac{1}{360}mdo + \frac{11}{310}en - \frac{1}{260}cq + \frac{1}{460}eo + \frac{5}{108}fns - \frac{1}{700}dmq + \frac{1}{300}ep + \frac{3}{1412}fos + \frac{8}{1308}lgn + \frac{1}{760}fpr + \frac{1}{270}go + \frac{9}{540}hn + \frac{1}{1426}lfq + \frac{17}{12420}gp + \frac{55}{9514}ho + \frac{6inr}{1425} + \frac{gq}{810} + \frac{5}{2180}hp + \frac{io}{570} + \frac{jn}{820} + \frac{4hq}{3180} + \frac{ip}{3210} + \frac{31jo}{4220} + \frac{3kn}{1544} + \frac{21iq}{4228} + \frac{51jp}{30820} + \frac{7kom}{1786} + \frac{5ln}{1104} + \frac{jqr}{570} + \frac{87kp}{2510} + \frac{17lo}{3720} + \frac{kqs}{628} + \frac{21lp}{4170} + \frac{7lqs}{5780}\right\}$$

Expression become complicated and further calculation to find focal values are not possible. For simplicity choose some coefficients zero

Theorem

Consider $C_{18,3}$, equations of type

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

with $\alpha(t), \beta(t)$ of degree 18 and 3 respectively.

$$\alpha(t) = a + dt^3 + bt + et^4 + ht^7 + \boldsymbol{s}t^{\mathbf{18}}$$

and

$$\beta(t) = u + yt3$$

Then $\mu_{max}\ C_{18,3} \geq 10$.

Proof

We compute $\eta_k\ k = 2,3,..,10$,focal values to calculate multiplicity of the origin. So polynomial we choose as coefficients of class we are discussing, using previous mentioned formulas

$$\eta_2 = u + \frac{y}{4}$$

$$\eta_3 = a + \frac{b}{2} + \frac{d}{4} + \frac{e}{5} + \frac{h}{8} + \frac{s}{19}$$

Now multiplicity of origin is two when $\eta_2 \neq 0$ and $\mu = 3$ when $\eta_3 is\ non\ zero$ or getting greater multiplicity select $\eta_2 and\ \eta_3$ both zero, which gives us,

$$u = -\frac{y}{4} \qquad (3.5.4)$$

$$a = -\frac{b}{2} - \frac{d}{4} - \frac{e}{5} - \frac{h}{8} - \frac{s}{19} \qquad (3.5.5)$$

We evaluate η_4 ,using above equations we obtain

$$\alpha(t) = -\frac{b}{2} - \frac{d}{4} - \frac{e}{5} - \frac{h}{8} - \frac{s}{19} + dt^3 + bt + et^4 + ht^7 + \boldsymbol{s}t^{\mathbf{18}} \qquad (3.5.6)$$

$$\beta(t) = -\frac{y}{4} + \frac{y}{4}t \qquad (3.5.7)$$

$$\eta_4 = \frac{1}{6292800} y(14580s + 15295h + 6992e - 26220b)$$

Now for $\eta_4 = 0$ i.e.

$$y(14580s + 15295h + 6992e - 26220b) = 0$$

Implies that either $y = 0$

or $\quad s = \left(-\frac{15295}{14580}h - \frac{6992}{14580}e - \frac{26220}{14580}b\right) \quad (3.5.8)$

If we avail possibility of $y = 0$,the result we get is $\beta(\text{t}) = 0$ also $\eta_3 = 0$ .Which shows that average values of alpha zero . Therefore by previously stated result the origin become center if we choose $y = 0$,so we neglect this possibility and avail other possibility that is

$$s = \left(-\frac{15295}{14580}h - \frac{6992}{14580}e - \frac{26220}{14580}b\right)$$

$\beta(\text{t}) = -\frac{y}{4} + \frac{y}{4}\text{t}$

using above equations, we calculate

$\eta_5 = y^2(\frac{21216b - 9740\,e - 17479\,h}{573168960})$

If $\eta_5 = 0$.

Then one possibility is

$h = \frac{21216b - 9740\,e}{17479} \quad (3.5.9)$

As we have no option to choose y=0

evaluate η_6 using

$\beta(\text{t}) = -\frac{y}{4} + \frac{y}{4}\text{t}$

$\eta_6 = \frac{y(5b + e)(526511745y^2 + 1037554880b - 73295288e)}{59098906907040000}$

We want $\eta_6 = 0$ which will be possible if

$y = 0$ and other option is

$b = \frac{-526511745y^2 + 73295288e}{1037554880} \quad (3.5.10)$

As cant choose the possibility of y=0 ,so other possibility is

$b = \frac{-526511745y^2 + 73295288e}{1037554880}$

To evaluate η_7 use

$\beta(\text{t}) = -\frac{y}{4} + \frac{y}{4}\text{t}$

$$\eta_7 = \frac{17181\,y^2(-15\,y + 8\,e)(-23436276854222315y^2 + 24713761516349472d + 17355654635081720e)}{213217536553985818810613760000`}$$

$\eta_7 = 0$ will be possible if y= 0 or

$$e = \frac{-1}{17355654635081720}(-23436276854222315y^2 + 24713761516349472d)$$

$$\eta_8 = (-5727p)(10342969708396957785p^2 + 2805011932105665072d)$$
$$(-7535915552939960490124827136486627905640680500793541701363685 75p^4 -$$
$$5435516740504153293788648619209342614442229666184472112583 80dp^2 +$$
$$846098474365028553989592008847580715746195397541577109437 92d^2\}$$
$$(\frac{1}{785420545155523182398698565700559638665845155781876056110487043469561339857858673\ldots}$$

$$\eta_9 = \frac{-p^5(3231695374879149027057862333421228337050+3537920022675940201967654484978249518889p)}{301503418083057169528674955861126461817651200}$$

$$\eta_{10} = \frac{-1850214428033743598569083569042606001992660866083570422496398313929 64645}{106833599277768940696058124987614991716}$$

= constant

So $\mu_{max}(C_{18,3}) = 10$ if all results hold and $y(s) \neq 0$.

3.4.2Theorem

Assess equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \tag{3.5.11}$$

$$\alpha(t) = \left(-\frac{3542964569707}{780345076192976}q^2 - \frac{1}{4}\varepsilon_2 - \frac{347887}{15325192}\varepsilon_1 - \frac{36}{5720}\varepsilon_3 - \frac{1}{7}\varepsilon_4\right) + \left(\frac{12426874}{2347697}q^2 - \varepsilon_1\right)t^2 + \left(\frac{1152461909}{348794090}q^2 + \varepsilon_3\right)t^7 + \left(-\frac{153399897}{778209}q^2 - \frac{95095}{47408}\varepsilon_3 + \varepsilon_4\right)t^8 + \left(\frac{8852}{130493}q^2 + \frac{81690}{693001}\varepsilon_2 + \varepsilon_3\right)t^{11}$$

$$\beta(t) = \left(\frac{552193173897193640}{1407090060591087} - \frac{1}{2}\varepsilon_1\right) + \varepsilon_8 + \left(-\frac{1104386347794387280}{1407090060591087} + \varepsilon_1\right)t$$

If $1 \leq p \leq 8$, if we assume these are non-zero satisfying condition each ε_p is smaller than ε_{p-1} then 8 real periodic solution that are non-trivial of above equation exist.

Proof

The origin multiplicity is 8 for selected above coefficients, if $\varepsilon_i = 0$ for $1 \leq i \leq 8$. Pick $\varepsilon_1 \neq 0$ but then unselected $\varepsilon_i = 0$ for $2 \leq i \leq 8$,also $\eta_2 = 0, \eta_3 = \dots\dots\dots\dots = \eta_5 = 0$, $\eta_7 \neq 0$ with η_7 is multiple of ε_i , so $\mu = 7$.

In this way multiplicity is equal to less than 1 of previous multiplicity. After this,choose $\varepsilon_2 \neq 0$ but $\varepsilon_3 = \varepsilon_4 = \dots\dots = \varepsilon_5 = 0$;and verify $\eta_2 = \eta_3 = \cdots . \eta_5 = 0$, $\eta_6 \neq 0$ and η_6 is constant multiple of ε_2, hence $\mu = 6$ obtained after minimize by one.

If value of ε_2 is between 0 and 1 then gives two real periodic solutions that are not trivial. By following this procedure, eight real periodic solutions also non-trivial are obtained.

Corollary

Using $\alpha(t)$, $\beta(t)$ as in above theorem,

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 + \gamma(t)z + \delta \tag{3.5.12}$$

Exhibits ten real periodic solutions in the limit of small γ and δ .

Proof

If $\gamma(t) = 0, \beta = 0$ with multiplicity 2 then above equation provide eight real periodic solutions. We choose $\gamma(t)$ is not equal to zero gives $\mu = 1$ then controversy used in previous theorem we attain real periodic nine solutions. As $z = 0$ also satisfy it, so we get real ten periodic solutions.

3.6 Resolution.

We calculate solutions that are periodic of certain classes but actual problem to derive a formula for Hilbert number is still unsolved

In short, the study of limit cycles is a complex area of research with its application to understand complex systems. The transformation (1.4.3) gives us a path for examining limit cycles, and the Poincaré-Bendixson theorem and bifurcation techniques are important methods to confirm the existence of limit cycles. Next We are interested center focus problem of higher order and make new methods to find behavior of limit cycle also find limit cycle application in new field of network science.

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