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Bifurcation of Periodic Solutions of First order non-Autonomous Differential Equations

Muhammad Irfan[1,] Mrs. Azra Aziz[2]

[1,2] Department of Mathematics and Statistics, Institute of Southern Punjab (ISP), Multan, Pakistan

Abstract

This article deals the development of the number of periodic solutions for ordinary differential equations.We investigate the focal values for first order non-autonomous differential equation for periodic solution from a fine focus .Periodic solutions with polynomial coefficients are executed for classes $C_{11,4}$, $C_{11,5}$ and $C_{11,6}$.Limit cycles are found .We present a variety of polynomial classes along with their bifurcation analysis which confirms the generality and authenticity of the method presented

Introduction

David Hilbert gave 23 problems [1] to the second international congress of mathematicians in Paris on August 8, 1900. The sixteenth issue was subject of geography of arithmetical bends and surfaces . It has two sections. Hilbert proposed a point by point investigation of relative places of various parts of arithmetical bends in nth-request vector fields in initial segment. In second part, Hilbert mentioned upper bound of number of limit point not really set in stone and their relative situations in n-dimensional polynomial vector fields. This sort of issue is pertinent to dynamical frameworks and common differential conditions.

In Hilbert's sixteenth problem, limit cycle hypothesis assumes significant part. The confounded contributor to issue is deciding number of limit point cycles for differential conditions. Poincaré previously found and presented limit cycle wonder in his four-section article fundamental bends portrayed by differential conditions [2-5], which was distributed somewhere in the range of 1881 and 1886. Poincaré additionally found that examination of cutoff cycles and arrangements of worldwide primary issues of a group of essential bends of differential conditions had a cozy relationship.

Bendixson later stretched out his work to incorporate notable Poincaré-Bendixson hypothesis [6] on the limit point set number of directions in limited space of dynamical frameworks. The development of triode vacuum tube, which could create consistent self-invigorated motions of steady adequacy, was the spurring power behind the investigation of breaking point cycle hypothesis. It was noticed that straight differential conditions don't satisfactorily clarify this type of swaying. Van der Pol toward the finish of 1920's, clarify the motions by built a differential condition for a triode vacuum tube with a steady adequacy. A wide range of dynamical frameworks have normal arrangements. They are a gathering of self-supporting motions in model frameworks. Additionally, as such, even without outside impacts these cycles sway. For instance, consider the Holing-Tanner hunter prey model [30]. This model will in general be seen in very much like hunter prey species e.g., in house sparrows and other regular creatures, for instance, Europe's sparrow peddles, the Central North's muskrat and mink, in Ontario, Canada, white-followed deer and wolves can be found. The beat of a heart, internal heat level rhythms, chemical emission, suddenly swaying synthetic responses, and vibration in extensions and plane wings are on the whole instances of self-energized wavering. Breaking point cycle hypothesis has been concentrated by physicists, and all the more as of late by scientific experts, researcher, and architects, because of inescapable event of cutoff cycles in science and innovation. The differential condition of structure is thought of.. The differential equation of form is considered.

$$\frac{dz}{dt} = s_0(t)z^3 + s_1(t)z^2 + s_2(t)z + s_3(t) \tag{1.1.1}$$

In above equation z is complex and q_i's real valued functions.
This form (1.1.1) is like as

$$\frac{dz}{dt} = s_0(t)z^n + s_1(t)z^{n-1} + \cdots + s_N(t) \tag{1.1.2}$$

Some attention within literature has been observed by such a class of equations with $s_0(t) = 1$. Differential equation's types (1.1.1) appeared in studies of Abel's differential equation [7-9]. This type of differential equation is of particular interest due to reference to Hilbert's Sixteenth problem for polynomial differential system of equations.

$$\dot{x} = \frac{dx}{dt} = X\ (x,y)$$
$$\dot{y} = \frac{dy}{dt} = Y\ (x,y)$$

X and Y are polynomials in x and y.
A limit cycle is periodic orbit isolated from all periodic orbits of planar differential system.
We have a premium inside second piece of Hilbert's Sixteenth's concern which is about most extreme number of cutoff patterns of framework (1.1.3) which can bifurcate from beginning or number of occasional arrangements which can be produced from z=0 under annoyance of coefficient for condition (1.1.1). We'll clarify Hilbert's Sixteenth issue for condition (1.1.1) in segment (1.2).
It was displayed in [5] and [13] that when q_0 (t)=1. Then, at that point Abel's condition (1.1.1) has assortment is intended for precisely three intermittent arrangements. In any case, neighborhood question related to Hilbert's Sixteenth issue is decreased to polynomial differential condition in which q_0 (t) has zeros.
For this case results of [23], for no additional time hold; indeed, Lins Neto [13] has been examined an illustration of the quantity of intermittent arrangement choices for condition (1.1.1) which confirm that there exist no upper bound except if a few restrictions are forced on the coefficients. Consequently, we thought about various classes in part 3 of thesis.
In meeting of congress of mathematician which was held in Paris in 1900 Hilbert's [5] give a list of 23 problems. Sixteenth problem was concerned with the system of type

$$\frac{dx}{dt} = R(x,y) \tag{1.2.1}$$
$$\frac{dy}{dt} = S(x,y)$$

in above equations R and S are n^{th} degree polynomials. Hilbert's sixteenth problems are used to determine the number of limit cycles which given system can have in terms of n and their possible relative positions. More formally let $\pi(p,q)$ enumerate limit cycle of system (1.2.1), and we define

$$H_n = sup\ sup\ \{\pi(r,s): degreeR, degreeS \leq n\} \tag{1.2.2}$$

Presently Hilbert's concern is utilized to give end to H_n in term of n and simultaneously we get data about the conceivable setup of cutoff cycle. Such sort of issue has ended up being undeniably challenging. The most extreme conceivable number of breaking point cycle is yet not known, in any event, for the case n=2. The essential job to explore Hilbert's Sixteenth issue was in 1952, when Bautin ended up being out of a balance point by bifurcating three cutoff periods, that is H_2≥3 [5]. In the year of 1955 Petroviskii and Landis claimed that $H_2 = 3$.
In the extended time of 1955 Petroviskii and Landis asserted that H_2=3. In any case, in no time a while later they pulled out their confirmations. It had been until 1979 when the instances of the

quadratic framework with something like four breaking point cycles were created by Shi and different mathematicians, subsequently H_2≥4. At the point when P and Q are symmetric cubic polynomials, H_3=5 it was displayed in reference [23] that five is most extreme number of cutoff cycles that can bifurcate out of beginning. For general cubic framework it is presently referred to that upwards of eight breaking point cycles can bifurcate. Concerning assessing H_3 without limitation of little adequacy limit cycles, N.G Lloyed have shown that H_3≥11 by considering bifurcation of breaking point cycles for certain Hamiltonian framework [23-25].

In 1923, Dulac proposed that such polynomial framework couldn't have proceeding with limit cycles. His evidence was observed to be inadequate and is as of late that Dulac's hypothesis has been demonstrated.

A couple of years prior, Boman showed that quadratic framework can't have proceeding with limit cycles and presently French mathematicians, Ecall, Martinet, Mussy and Ranis have given a proof in everyday case. They utilized some refined thoughts concerning non focalized power series (1.2.1) are frequently composed as

$$\frac{dx}{dt} = \text{ƛ}x + y + R_n(x,y)$$
$$\frac{dy}{dt} = -x + \text{ƛ}y + S_n(x,y) \qquad (1.2.3)$$

It can be transformed to non-autonomous equation

$$\frac{d\rho}{d\theta} = \text{Ϭ}(\theta)\rho^3 + \text{ϐ}(\theta)\rho^2 - \text{ƛ}(n-1)\rho \qquad (1.2.4)$$

using the transformation (1.4.3), which we will explain in section (1.4). Note that limit cycles of (1.2.3) assimilate to 2π periodic solution of (1.2.4). Hence Hilbert's problem is now to seek out the 2π periodic solution of (1.2.4). Therefore, we have an interest in this thesis to seek out periodic solutions of equation (1.2.4). For n=2 Lins Neto [13] has proved the following proposition.

1.2 Proposition

The sixteenth problem of Hilbert for degree two equation in plane can be reduced to the problem of finding the maximum number for closed solution of the equation.

$$\frac{d\rho}{d\theta} = \text{Ϭ}(\theta)\rho^3 + \text{ϐ}(\theta)\rho^2 - \text{ƛ}(n-1)\rho$$
$$0 \le \theta \le 2\pi$$

where $\text{Ϭ}(\theta)$ and $\text{ϐ}(\theta)$ are homogeneous polynomials in $cos(\theta)$ and $sin(\theta)$ of degree six and three respectively, ƛ is a constant.

Indeed, Lins Neto [13] has given example which demonstrates there exist no upper bound for the number of periodic solutions of (1.4.4) until, that the constant number (coefficient) is properly limited. The important answerable inquiry is that the many achievable number of periodic solutions for different classes of coefficients. For the existence of periodic solutions, we'd need Poincare Bendixson Theorem which we are going to state in next section.

1.3 Poincare Bendixson theorem

"In the event that the framework yielded (1.1.1) has an answer that stays in a limited district and doesn't arrive at a state of balance, then, at that point either the arrangement is an intermittent circle itself or it twistings towards an occasional arrangement".

Henceforth it is feasible to show legitimacy of cutoff cycles by exhibit a limited locale inside directions be left and that doesn't have a balance point. It goes next that a particularly limited area will not be effectively joined on the grounds that a cutoff cycle should have a harmony point in its inside.

The second strategy for demonstrating presence of breaking point cycles includes some sort of bifurcation procedures. "Bifurcation" is a marvel related with trade of arrangement security on account of coefficients of X and Y are shifted to such an extent that an answer switches steadiness, then, at that point when the arrangements lose its strength, it branches into at least one arrangements. The quantity of cutoff cycles which bifurcate from a fine spotlight relies upon how close the basic point be a middle since then bifurcation stops and bifurcated cutoff cycles vanish. Along these lines there is a necessity for tests for a basic highlight be a middle which we will give in next section (See additionally [15]).

In the following area we are having the chance to talk about the change which we used to change the framework (1.1.1) to non-independent first request differential condition.

$$\frac{dz}{dt} = 6(t)z^3 + ϐ(t)z^2 + \gamma(t) \tag{1.3.1}$$

and the connection between (1.3.1) and

$$\frac{dx}{dt} = ƛx + y + R_n(x, y)$$
$$\frac{dy}{dt} = -x + ƛy + S_n(x, y)$$

is discussed in next section.

1.4 Polynomial system and Abel's equation

We have an interest to determine the number of periodic solutions of equation (1.2.4). We all know that system (1.2.1) can be written as

$$\frac{dx}{dt} = ƛx + y + R_n(x, y) \tag{1.4.1}$$
$$\frac{dy}{dt} = -x + ƛy + S_n(x, y)$$

In above equation R_n and S_n were also degree n homogeneous polynomials.

We use transformation

$$T: (\mathrm{r}, \theta) \longrightarrow (\rho, \theta) \tag{1.4.2}$$

where

$$\rho = \mathrm{r}^{n-1}\left(1 - \mathrm{r}^{n-1}g(\theta)\right)^{-1} \tag{1.4.3}$$

It is defined in an open set $D = \{(\mathrm{r}, \theta), \mathrm{r}^{n-1}g(\theta) < 1\}$ containing the origin.

This transformation has been used in a number of investigations [15] and [21] in polar coordinates system (1.4.1) becomes

$$\frac{dr}{dt} = \lambda\mathrm{r} + \mathrm{r}^n f(\theta), \frac{d\theta}{dt} = 1 - \mathrm{r}^{(n+1)}(g(\theta)) \tag{1.4.4}$$

where $f(\theta), g(\theta)$ are polynomials in $cos\theta$ and $sin\theta$ of degree n + 1.

In the $case\ n = 2$ the transformation (1.2.2) appears to have been exploited firstly by Lins Neto [13]. Differentiating (1.4.3) w.r.t θ, we get

$$\frac{d\rho}{d\theta} = 6(\theta)\rho^3 + ϐ(\theta)\rho^2 - ƛ(n-1)\rho \tag{1.4.5}$$

where

$$6(\theta) = -(n-1)g(\theta)\{f(\theta) + ƛg(\theta)\}$$
$$ϐ(\theta) = -(n-1)\{f(\theta) + 2ƛg(\theta) + g(\theta)\}$$

From the equation (1.4.6) it is clear that the transformation (1.4.3) can be written as

$$\mathrm{r}^{n-1} = \frac{\rho}{1 + \rho g(\theta)} \tag{1.4.6}$$

The transformation maps $r = 0$ to $\rho = 0, \mathrm{r} > 0$ to $\rho > 0$ and a neighborhood of $\mathrm{r} = 0$ to neighborhood of $\rho = 0$, provided $\rho > 0$ and $1 + \rho g(\theta) > 0\ \forall\ \theta$. It is simple to verify that the system given in (1.4.2), the point of equilibrium at the origin relate to the constant solution $\rho = 0$ of (1.4.5) and low frequency limit cycles of (1.4.1) relate to periodic solution of 2π of (1.4.5) with ρ small and positive.

2-Periodic solutions generation process

We start this segment by thinking about a non-independent differential equation of first order of the given kind

$$\frac{dz}{dt} = Ϭ(t)z^3 + ϐ(t)z^2 + \gamma(t) \qquad (2.1.1)$$

in above condition z is complex yet t stay as vaiable and the Ϭ, ϐ and γ are real coefficients . To look for data identified with number of arrangements we fixed the ω ∈R which satisfy the necessities of intermittent limit condition.

$$z(0) = z(\omega) \qquad (2.1.2)$$

The coefficients given in condition (2.1.1) are occasional if the Z is complex. We have particularly interested in the circumstance when we select ω =1 and α and β are polynomials in t. Equation (2.1.1) shown that it has various occasional arrangements as the given genuine condition.

$$\frac{d\rho}{d\theta} = Ϭ(\theta)\rho^3 + ϐ(\theta)\rho^2 - ƛ(n-1)\rho$$

Our first concern is to discover the way in the examination of number of intermittent arrangements of the above condition (2.1.1) is to bode well which arrangement are developed or destructed, focal interest in our work is bifurcation. We need to make upper limit by which a given arrangement can bifurcate for given number of periodic solutions.

It has been seen in [15] that an answer may parts into all things considered μ remarkable periodic solutions that has assortment μ. To figure the assortment of beginning z = 0 , the cycle that we utilized is straightforward and discussed in [13 - 15] . To accomplish fulfillment, we'll portray it in next section (2.2).

2.2 Computation of the beginning's assortment

The assortment of z = 0 as the arrangement of

$$\frac{dz}{dt} = Ϭ(t)z^3 + ϐ(t)z^2 + \gamma(t) \qquad (2.2.1)$$

is the multiplicity of $z = 0$ as the zero of the displacement function.

$$Ձ_p; c \longrightarrow z_p(\omega; 0, c) - c$$

Above function already have a definition and for an open set containing the origin is holomorphic. Now we want to explain a method which is used to compute the multiplicity of $z = 0$ for $0 \le t \le \omega$ and c in the neighboring of 0, we should write $z_p(t, 0, c)$ as a power series.

$$z_p(t, 0, c) = \sum_{n=1}^{\infty} ą_n(t)c^n$$

where

$$ą_1(0) = 1 \text{ and } ą_n(0) = 0 \text{ if } n > 1$$

Thus

$$Ձ_p(c) = z_p(\omega, 0, c) - c$$
$$= \sum_{n=1}^{\infty} ą_n(\omega)c^n - c$$
$$Ձ_p(c) = (ą_1(\omega) - 1)c + \sum_{n=1}^{\infty} ą_n(\omega)c^n \qquad (2.2.2)$$

From above it is clear that the multiplicity of $z = 0$ is $\mu > 1$ if

$ą_1(\omega) = 1, ą_2(\omega) = ą_3(\omega) = ą_4(\omega) = \cdots\cdots = ą_{\mu-1}(\omega) = 0$ and $ą_\mu(\omega) \neq 0$

Origin become a center when

$ą_1(\omega) = 1$ and $ą_k(\omega) = 0$ for all $k > 1$

The equation given in (2.2.2) substitute into the equation given (2.2.1). To doing this we get conceptual set of linear differential equation for $ą_n(t)$; the initial conditions are $ą_1(0) = 1, ą_j(0) = 0$, for $j > 1$

It may be note that

$$\dot{a}_1(t) = ą_1(t)\,\gamma(t)$$

where

$$ą_1(t) = exp\left(\int_0^{\omega} \gamma(s)ds\right)$$

Therefore, $\mu > 1$ iff

$$\int_0^{\omega} \gamma(s)ds = 0 \tag{2.2.3}$$

Since our main concern in that situation when $z = 0$ become a multiple alternative solution, we'll suppose that (2.2.3) remain in such state. Into the transformation

$$\xi = z\, exp\left[-\int_0^{1} \gamma(s)ds\right] \tag{2.2.4}$$

equation (2.2.1) becomes

$$\dot{\xi} = \hat{\alpha}(t)\xi^3 + \hat{\beta}(t)\xi^2 \tag{2.2.5}$$

where

$$\hat{\alpha}(t) = б(t)exp\left(2\int_0^{t} \gamma(s)ds\right)$$

and

$$\hat{\beta}(t) = ϐ(t)exp\left(\int_0^{t} \gamma(s)ds\right)$$

It is clearly shown that, if the functions α, β and γ that are given n above equation are periodic, then б and ϐ in the above equation are also periodic.

If the multiplicity of $z = 0$ as a periodic solution of (2.1.1) is $\mu > 1$, as shown in [15]. Then the multiplicity of $\xi = 0$ is also μ as a periodic solution of (2.2.5), so we conclude that $\gamma(t) = 0$, and conclude an equation of the form

$$\frac{dz}{dt} = б(t)z^3 + ϐ(t)z^2 \tag{2.2.6}$$

For the above equation (2.2.6), $ą_1(t) = 1$ and, for $n > 1$, the functions $ą_n(t)$ are demonstrated by the relation

$$\dot{a}_n = б\sum^{ą}_{\substack{i+j+k=n\\ i,j,k \ge 1}} ą_ią_ją_k + ϐ\sum^{ą}_{\substack{i+j=n\\ i,j \ge 1}} ą_ią_j \tag{2.2.7}$$

The above equations may have solution with recursively; the calculating work may turn into complication as we increase the value of n because the calculating process is tedious and have integration by parts. We limit our focus therefore to $n \le 9$.

Let

$$ɳ_1 = ą_1(\omega)$$

The focal values of $\eta_i's$ were named by us as $i = 1,2,3,\ldots,8,9$. The definition is then given as

Definition (2.2.1)

The multiplicity of $z = 0$ that we express as μ is k if

$ɳ_1 = 1$ and $ɳ_2 = ɳ_3 = \ldots\ldots\ldots = ɳ_{k-1} = 0$ but $ɳ_k \neq 0$

The formulae for $ą_k(t)$ and for $k \le 9$ are given in [15] and for k ≤ 10 given in [26-28]. Now in section (2.3) we describe the focal values, because we use them in chapter 3.

2.3 Computation of $ą_j(t), ɳ_j(t)$ and the perturbation procedure

The theorem that is given below we give $ą_j(t)$ and $ɳ_j(t)$ for $j \le 10$ (for detail see [26-28]. To indicate an infinite integral that is a bar over a function

$$\underline{б(t)} = \int_0^{\omega} б(s)ds.$$

Theorem (2.3.1)

Equation of $\frac{dz}{dt} = б(t)z^3 + ϐ(t)z^2$, the function $ą_2, ą_3, \ldots\ldots, ą_{10}$ are given by the following formula.

$ą_2 = \underline{ϐ}$

$ᶏ_3 = \underline{ϐ}^2 + \underline{6}$
$ᶏ_4 = \underline{ϐ}^3 + 2\underline{ϐ}6 + \underline{ϐ6}$
$ᶏ_5 = \underline{ϐ}^4 + 3\underline{ϐ}^2\underline{6} + \underline{\underline{ϐ}^2 6} + 2\underline{ϐϐ6} + \frac{3}{4}\underline{6^2}$
$ᶏ_6 = \underline{ϐ}^5 + 4\underline{ϐ}^3\underline{6} + \underline{\underline{ϐ}^3 6} + 3\underline{ϐ}^2\underline{6\alpha} + 2\underline{ϐ\underline{ϐ^2 6}} + \frac{9}{2}\underline{ϐ6^2} + 3\underline{\underline{ϐ}6}\,\underline{6} - \frac{1}{2}\underline{ϐ\underline{6}^2}$
$ᶏ_7 = \underline{ϐ}^6 + 5\underline{ϐ}^4\underline{6} + \underline{\underline{ϐ}^4 6} + 4\underline{ϐ}^3\underline{ϐ6} + 2\underline{\underline{ϐ}^3 6}\,\underline{ϐ} + \frac{17}{2}\underline{ϐ}^2\underline{\alpha^2} + 3\underline{\underline{ϐ}^2\alpha}\,\underline{\alpha} + 2\left(\underline{ϐ\alpha}\right)^2 + 2\underline{\underline{ϐ}^2\underline{\alpha}}\alpha +$
$8\underline{\underline{ϐ}\alpha}\,\underline{ϐ\alpha} - \underline{ϐ\underline{ϐ\alpha^2}} + \frac{5}{2}\underline{\alpha}^3$
$a_8 = \underline{ϐ}^7 + 6\underline{ϐ}^5\underline{\alpha} + \underline{\underline{ϐ}^5\alpha} + 5\underline{ϐ}^4\underline{ϐ\alpha} + 2\underline{\underline{ϐ}^4\alpha}\,\underline{ϐ} + 4\underline{ϐ}^3\underline{\underline{ϐ}^2\alpha} + 3\underline{\underline{ϐ}^3\alpha}\,\underline{\alpha} + 3\underline{\underline{ϐ}^3\alpha}\,\underline{ϐ}^2 + 3\underline{\underline{ϐ}^3\alpha\underline{\alpha}} +$
$\frac{27}{2}\underline{ϐ}^3\underline{\alpha}^2 - \frac{3}{2}\underline{ϐ}^2\underline{ϐ\underline{\alpha}^2} + 15\underline{ϐ}^2\underline{ϐ\alpha}\,\underline{\alpha} + 4\underline{\underline{ϐ}^2\underline{\alpha}\alpha}\,\underline{ϐ} + \underline{\underline{ϐ}^2\alpha\bar{\beta}\alpha} + 12\underline{\underline{ϐ}^2\alpha\underline{ϐ}\alpha} + 8\underline{\underline{ϐ}^2\alpha ϐ\underline{\alpha}} + 5\left(\underline{ϐ\alpha}\right)^2 -$
$\frac{1}{2}\underline{ϐ\underline{\alpha}^2} - \frac{3}{2}\underline{ϐ\underline{\alpha}^2}\,\underline{\alpha} + 10\underline{ϐ\alpha}^3$
$a_9 = \underline{ϐ}^8 + 7\underline{ϐ}^6\underline{\alpha} + \underline{\underline{ϐ}^6\alpha} + 6ϐ^5\underline{ϐ\alpha} + 2\underline{\underline{ϐ}^5\alpha}\,\underline{ϐ} + 5\underline{ϐ}^4\underline{\underline{ϐ}^2\alpha} + 3\underline{\underline{ϐ}^4\alpha}\,\underline{\alpha} + 3\underline{\underline{ϐ}^3\alpha}\,\underline{ϐ}^2 + 5\underline{\underline{ϐ}^4\alpha\underline{\alpha}} +$
$\frac{39}{2}\underline{\beta}^4\underline{\alpha}^2 - 2\underline{\beta}^3\underline{\beta\underline{\alpha}^2} + 24\underline{\beta}^3\underline{\underline{\beta}\alpha}\,\underline{\alpha} + 6\underline{\underline{\beta}^3\alpha\underline{\alpha}}\,\underline{\beta} - 10\underline{\underline{\beta}^3\alpha\bar{\beta}\alpha} + 12\underline{\beta}\,\underline{\alpha}\,\underline{\underline{\beta}^3\alpha} + 4\underline{\underline{\alpha}\beta\underline{\beta}^3\alpha} + 4\underline{\underline{\beta}^3\alpha\underline{\beta}^3} +$
$\frac{43}{6}\underline{\beta}^2\underline{\alpha}^3 + 4\underline{\underline{\alpha}^3\beta\underline{\beta}} + 4\underline{\beta}^2\underline{\alpha}\,\underline{\alpha\underline{\beta}^2} - 10\underline{\underline{\beta\underline{\beta}\,\underline{\alpha}\,\underline{\alpha\underline{\beta}^2}}} + \frac{15}{2}\underline{\alpha}^2\underline{\underline{\beta}^2\alpha} + 2\underline{\beta}^2\underline{\underline{\beta}^2\alpha} - 2\underline{\beta}^4\underline{\alpha} + 8\underline{\underline{\alpha}\beta\underline{\beta}^3} +$
$2\underline{\underline{\beta\beta}^2\alpha\underline{\underline{\beta}\alpha}} + 26\underline{\underline{\underline{\beta}\alpha\underline{\beta}^2\alpha\underline{\alpha}}} + 6\underline{\underline{\beta}^2\alpha\underline{\alpha}} - 6\underline{\underline{\underline{\beta}}^2\alpha\underline{\alpha}} + 12\underline{\beta}^2\underline{\underline{\beta}\alpha\underline{\alpha}} + 16\underline{\underline{\beta}^2\alpha}\beta\underline{\beta}\,\underline{\alpha} - 16\underline{\underline{\beta}^2\alpha\underline{\underline{\beta}}\alpha} +$
$9\underline{\beta}^2\underline{\left(\underline{\beta}\alpha\right)}^2 + 9\underline{(\underline{\beta}\alpha)^2}\,\underline{\alpha} - \underline{\beta\underline{\alpha}^3}\,\underline{\beta} + \frac{35}{8}\underline{\alpha}^4 - 6\underline{\beta}\,\underline{\alpha}\,\underline{\beta\underline{\alpha}^2} + 6\beta\underline{\underline{\beta}^4\alpha\underline{\alpha}} + 33\underline{\beta}\,\underline{\alpha}^2\underline{\underline{\beta}\alpha} - 24\beta\underline{\underline{\underline{\alpha}^2\underline{\underline{\beta}}\alpha}} +$
$6\underline{\underline{\underline{\beta}}^2\underline{\alpha}\alpha\underline{\alpha}} - 4\underline{\beta\underline{\alpha}\,\underline{\beta}\,\underline{\alpha}^2}$
$a_{10} = \underline{\beta}^9 - \frac{23}{2}\underline{\beta}^7\underline{\alpha} - \frac{1235}{6}\underline{\underline{\beta}^5\underline{\alpha}\alpha} + 3\underline{\underline{\beta}^5\alpha\underline{\alpha}} + 111\underline{\beta}^4\underline{\underline{\beta}\alpha} - 444\underline{\underline{\alpha}\beta\underline{\beta}^3\underline{\underline{\beta}}\gamma} + 20\underline{\alpha}\,\underline{\beta\underline{\beta}^4\alpha} -$
$12\underline{\underline{\alpha}\beta\underline{\underline{\underline{\beta}^4\alpha}}} + \frac{214}{3}\underline{\alpha}\,\underline{\beta}^3\underline{\underline{\beta}^2\alpha} - 160\underline{\underline{\alpha}\beta\underline{\beta}^2\underline{\underline{\underline{\beta}^2\alpha}}} + \frac{15}{2}\underline{\alpha}^2\underline{\underline{\beta}^3\alpha} - \frac{970}{3}\underline{\alpha\underline{\alpha}^2\underline{\beta}^3} + 30\underline{\alpha}\,\underline{\beta}^2\underline{\beta^3\alpha} -$
$68\underline{\underline{\alpha}\beta\underline{\beta\beta\underline{\underline{\beta^2\alpha}}}} + 9\underline{\alpha}\,\underline{\underline{\beta}^3\alpha\underline{\alpha}} + \frac{1015}{9}\underline{\beta}^3\underline{\alpha}^3 - 237\underline{\beta\underline{\beta}^2\underline{\alpha}^3} - \frac{11}{2}\underline{\alpha}\,\underline{\beta}^2\underline{\underline{\beta}\,\underline{\alpha}^2} + 26\underline{\underline{\alpha}\beta\underline{\underline{\beta\beta\alpha^2}}} +$
$\frac{319}{2}\underline{\alpha}\,\underline{\underline{\underline{\beta}^3\alpha\underline{\alpha}}} - 174\underline{\underline{\beta\underline{\beta}\,\underline{\alpha}^2\underline{\underline{\beta}\alpha}}} - 90\underline{\underline{\alpha}\alpha\underline{\beta\underline{\underline{\beta^2\alpha}}}} + 24\underline{\alpha}\,\underline{\underline{\beta}^2\alpha\beta\underline{\alpha}} + 40\underline{\alpha}\,\underline{\beta}\,\underline{\alpha\underline{\alpha}\,\underline{\beta}^2} - 24\underline{\alpha}\beta\underline{\alpha\underline{\alpha}\,\underline{\beta}^2} +$
$3\underline{\alpha}\,\underline{\underline{\beta}^2\alpha\underline{\underline{\beta}\alpha}} - 154\underline{\underline{\alpha}\alpha\underline{\beta}^2\underline{\underline{\beta}\alpha}} - 24\underline{\alpha\underline{\alpha}^2\underline{\beta}^2\beta} + 70\underline{\alpha}\,\underline{\beta}\left(\underline{\underline{\beta}\alpha}\right)^2 + 42\underline{\underline{\alpha}\beta\left(\underline{\underline{\beta}\alpha}\right)^2} - 70\underline{\alpha}\,\underline{\underline{\beta}^3\alpha^2} -$
$\frac{3}{2}\underline{\alpha}\,\underline{\beta\underline{\alpha}^2} - 21\underline{\beta\underline{\alpha}^4} - \frac{15}{4}\underline{\alpha}^2\underline{\beta\underline{\alpha}^2} + \frac{169}{4}\underline{\alpha}^4\underline{\beta} + 24\alpha\underline{\alpha}^2\underline{\beta\underline{\beta^2\alpha}} - 24\underline{\underline{\alpha}^2\beta\underline{\underline{\beta}^2\alpha}} + 10\underline{\alpha}^3\underline{\underline{\beta}\alpha} + \frac{9}{2}\underline{\beta}^4\underline{\underline{\beta}^3\alpha} +$
$8\underline{\alpha}\,\underline{\beta}^7 - 74\alpha\underline{\underline{\underline{\alpha}}^3\underline{\beta}} + 8\underline{\beta}\,\underline{\beta\underline{\beta}\,\underline{\alpha}^3} + \frac{34}{3}\underline{\alpha}\,\underline{\beta}^3\underline{\underline{\beta}^2\alpha} + 2\underline{\beta\underline{\beta^6\alpha}} + 7\underline{\beta}^6\underline{\underline{\beta}\alpha} + 3\alpha\underline{\beta}^7 - 5\underline{\alpha\underline{\beta}^6} + 6\underline{\beta}^5\underline{\underline{\beta}^2\alpha} -$
$6\underline{\alpha\underline{\beta\beta^4}\alpha} + 2\underline{\beta}^3\underline{\underline{\beta}^3\alpha} + 10\underline{\beta}\,\underline{\alpha\underline{\alpha}\beta^4} + 26\underline{\alpha}^2\underline{\beta}^5 - \frac{5}{2}\underline{\beta}^4\underline{\beta\underline{\alpha}^2} + \frac{5}{2}\underline{\beta\underline{\beta}^4\underline{\alpha}^2} + \frac{73}{2}\underline{\alpha}\,\underline{\beta}^4 - \frac{127}{2}\underline{\underline{\beta}^4\alpha\underline{\underline{\beta}\alpha}} +$
$9\underline{\beta}^2\underline{\alpha\underline{\alpha}\,\underline{\beta}^2} - 20\underline{\beta\underline{\beta^3\alpha\underline{\underline{\beta}\alpha}}} + 19\alpha\underline{\alpha}^2\underline{\underline{\beta}^3\alpha} - 21\underline{\underline{\beta}^2\alpha\underline{\underline{\beta}^3\alpha}} + 8\underline{\beta}\,\underline{\underline{\alpha}\beta\underline{\beta}^3\alpha} - \frac{160}{3}\underline{\alpha\underline{\beta}^3\underline{\underline{\beta}^2\alpha}} +$
$32\underline{\underline{\beta}^4\alpha\underline{\underline{\beta}\alpha}} - 20\underline{\beta}\,\underline{\alpha\underline{\beta}\beta\underline{\underline{\beta}^2\alpha}} + 24\underline{\beta}\,\underline{\alpha}^2\underline{\underline{\beta}^2\alpha} + \frac{4}{3}\underline{\beta}^3\underline{\underline{\beta}^2\alpha} - \frac{31}{30}\underline{\alpha\underline{\beta}^5} - \frac{4}{5}\underline{\alpha}\,\underline{\beta}^5 + 16\underline{\beta\underline{\beta^3\underline{\alpha}\beta}} -$
$16\underline{\underline{\alpha}\beta\underline{\beta}^4} + 3\underline{\beta}^2\underline{\underline{\beta}^2\alpha\underline{\underline{\beta}\alpha}} + 42\underline{\beta}^2\underline{\underline{\beta}^2\alpha\underline{\beta}\alpha} + 12\underline{\alpha}\,\underline{\beta\underline{\beta^2\alpha}} + 12\underline{\alpha\underline{\beta\beta}^2\alpha} - 12\underline{\beta\underline{\beta^2\underline{\alpha}\alpha}} + 12\underline{\beta}^3\underline{\underline{\beta}^2\underline{\alpha}\alpha} +$
$32\underline{\beta^2\underline{\alpha}\,\underline{\beta\beta}^2\alpha} - 32\underline{\beta\underline{\beta^3\alpha\underline{\beta}\,\underline{\underline{\alpha}}}} + 14\underline{\beta}^3\left(\underline{\underline{\beta}\alpha}\right)^2 + 15\underline{\underline{\beta}^5\alpha^2} - 28\alpha\underline{\beta}\left(\underline{\underline{\beta}\alpha}\right)^2 - \frac{3}{2}\underline{\beta}^2\underline{\beta\underline{\alpha}^3} +$

$$\frac{13}{2}\underline{\underline{\beta}^2\alpha\beta\underline{\underline{\alpha}}^2}+12\underline{\beta}\underline{\beta\underline{\alpha}^2\beta\underline{\alpha}}-36\underline{\beta\underline{\beta}\underline{\alpha}^2\beta\underline{\alpha}}-48\underline{\beta}\underline{\underline{\alpha}^2\beta\underline{\underline{\beta}}\alpha}-16\underline{\alpha\underline{\beta}\underline{\alpha\underline{\alpha}\underline{\beta}^2}}-8\beta\underline{\beta}\underline{\underline{\alpha}\beta\underline{\underline{\alpha}}^2}+$$

$$2\underline{\beta}^3\underline{\underline{\beta}^4\alpha}+8\underline{\beta}^2\underline{\underline{\beta}^2\alpha\beta\underline{\alpha}}-8\beta\underline{\underline{\beta}^4\alpha\underline{\alpha}}-2\underline{\alpha\underline{\beta}\underline{\underline{\beta}^4\alpha}}+\frac{1}{2}\underline{\beta}^4\underline{\underline{\beta}\alpha}+2\beta\underline{\underline{\beta}\alpha\underline{\beta}^3\alpha}+\underline{ϐ\underline{\underline{ϐ}}\alpha ϐ\underline{\alpha}^2}+\underline{ϐ}\left(\underline{\underline{ϐ}^2\alpha}\right)^2$$

Now we assume the result which given below help us to calculating the multiplicity.

Theorem (2.3.2)

The solution $z = 0$ where

$$\frac{dz}{dt} = 6(t)z^3 + ϐ(t)z^2$$

there is a multiplicity of k is of $2 \leq k \leq 10$ iff $\eta_1 = 0$ for $2 \leq j \leq k - 1$ and $\eta_k \neq 0$ in which

$\eta = \int_0^\omega ϐ\, dt$

$\eta_3 = \int_0^\omega 6\, dt$

$\eta_4 = \int_0^\omega 6\,\underline{\beta}\, dt$

$\eta_5 = \int_0^\omega \quad 6\underline{ϐ}^2$

$\eta_6 = \int_0^\omega \quad (6\underline{ϐ}^3 - \frac{1}{2}\underline{6}^2ϐ)$

$\eta_7 = \int_0^\omega \quad (6\underline{ϐ}^4 + 26\underline{6ϐ}^2)$

$\eta_8 = \int_0^\omega \quad (6\underline{ϐ}^5 + 36\underline{6ϐ}^3 + 6ϐ^2\underline{\underline{ϐ}6} - \frac{1}{2}\underline{6}^3ϐ)$

$\eta_9 = \int_0^\omega \quad (6ϐ - 56\underline{6ϐ}^4 - 2\underline{ϐ}^3\underline{\underline{ϐ}6} + 20\underline{ϐ\underline{6}^2} + 2\underline{ϐ6}ϐ\underline{6}^2)$

and

$$\eta_{10} = \int_0^\omega \quad (6\underline{ϐ}^7 - \frac{1235}{6}6\underline{6ϐ}^5 - \frac{970}{3}ϐ6^2\underline{ϐ}^3 - 23766^2\underline{ϐ}^3 - 24ϐ6^2\underline{ϐ}^2 - 70\underline{6ϐ}^36^2 - 21\underline{6}^4ϐ$$
$$- 746\underline{6}^3\underline{ϐ} + \frac{5}{2}\underline{6}^26\underline{ϐ}^4 + 32\underline{ϐ}^46\underline{\underline{ϐ6}} - 16ϐ\underline{ϐ}^4\underline{6} - 15\underline{ϐ}^56^2 - 36ϐ\underline{ϐ6}^2\underline{\underline{ϐ6}} - 8ϐ\underline{ϐ}^46)$$

We need to make a condition that have a greatest attainable number of one of a kind genuine occasional not set in stone variety μ. The arrangement that we make for annoyance of coefficients, every one of them gives no less than one occasional bifurcation arrangement from of the beginning.

We consider an equation of the type given in (2.2.1) that $\mu = k$ suggest. We assume that a neighboring says U with a root throughout complex plane having no periodic solution except for $z = 0$. Then the theorem (2.4) in [15] shows that the total number of periodic solutions with initial points in U remains unchanged due to the minor disturbance of the coefficient. We perturb the coefficients6, $ϐ$ and γ, if relevant, because then

$$\eta_2 = \eta_3 = \ldots\ldots\ldots\ldots = \eta_{k-2} = 0 \text{ but } \eta_{k-1} \neq 0$$

Then we'll have a periodic solution ψ(t) suggest, that is non-trivial, with $\psi(0) \in U$ are ψ and the zero solutions. Already known complicated solutions appear in coupled combination which assume that ψ is real. Then W_1 be an adjacency of ψ and U_1, be an adjacency of $z = 0$ in a way that

$$U_1 \cup W_1 \subset \text{ and } U_1 \cap W_1 = \emptyset$$

In just one of $U_1$and W_1 the number of periodic alternatives has initial points is continued under sufficiently restricted perturbations of the coefficient we then make an effort to further disturb the coefficients in that way $\eta = \eta_3 = \ldots\ldots\ldots\ldots = \eta_{k-1} = 0$ but $\eta_{k-2} \neq 0$. In such situation $\mu = k - 2$. Now there is an initial point in U_1 for a second real non-trivial periodic solution, leaving a real periodic solution with an initial point in W_1. Therefore, we have two real periodic solutions that are non-trivial, as well as the zero solution is $K - 2$ multiplicity. We go forward in such order to get an equation form of (2.2.5) with $\mu = 2$ and $k - 2$ separate non-trivial real periodic solutions.

2.4 Criteria for a center

In order to find a number of achievable μ values for different groups of equations, we will evaluate the quantity of $ɳ_k = ą_k(\omega)$ (almost) given in the theorem (2.3.2). Unless a value k of K is found with the property that this process continues
$ɳ_k = 0$ for all if $ɳ_2 = ɳ_3 = \dots\dots\dots\dots = ɳ_{k+1} = 0$ then μ_{max} is minimum in these K.
We need circumstances that are necessary for $z = 0$ to be a center in accordance with the method we have mentioned for computing the $ɳ_k$. Only then we realise that we no longer need to measure theη_k. The conditions for $z = 0$ to be a centre, which are given in [15], are now listed here because we will need them in chapter 03 to find μ_{max}.

Theorem (2.4.1)
Assume that there is still a function ß which is differentiable from $ß(\omega) = ß(0)$ and function which are continuous that is s and t defined by $I = ß([0, \omega])$ that is

$$б(t) = f\big(ß(t)\big)\dot{ß},$$
$$ϐ(t) = g\big(ß(t)\big)\dot{ß}.$$

Then the origin $z = 1$ is center for the equation

$$\frac{dz}{dt} = б(t)z^3 + ϐ(t)z^2 \qquad (2.4.1)$$

Corollary (2.4.2)
Assume an equation (2.4.1) in which б is a constant multiple of ϐ and $\int_0^\omega \quad ϐ(t)dt = 0$
then the origin is a center as discussed in [16].
Corollary (2.4.3)
Assume that б and ϐ is identically zero and the other has mean value zero. Then the origin is a center as discussed in [16].
Corollary (2.4.4)
Assume that б and ϐ contains odd powers of $sin(t)$ or $cos(t)$. Then the origin is a center as discussed in [16].

2.5 Summary of some known results

Here let an equation of the form

$$\frac{dz}{dt} = б(t)z^3 + ϐ(t)z^2 \qquad (2.5.1)$$

The above equations were firstly used by Lins Neto in [13] where the equation take up with number of periodic solutions. A number of examples were given by him of equation (2.5.1) with coefficient ϐ has degree one and б has degree b_1, at least with $\frac{b_1}{2} + 3$ periodic solutions. Two types of coefficients of (2.5.1) were used by Alwash and Lloyd in [15] that (i) function of polynomial in t (ii) polynomiał function in $cos(t)$ and $sin(t)$ and explain examples of the number of periodic solutions that overcome the limits given by Lins Neto [13]. The class $C_{2,3}$ which was considered by Alwash we use the result here.
Theorem (2.5.1)
We consider the class $C_{2,3}$ which show the equation of the kind given below

$$\frac{dz}{dt} = б(t)z^3 + ϐ(t)z^2$$

in the above equation $б(t)$ has degree three and $ϐ(t)$ has degree two. That is,

$$ϐ(t) = a + bt + ct^2$$
$$б(t) = d + et + ft^2 + gt^3$$

then $\mu_{max}\ C_{2,3} = 8$.
The class $C_{k,1}$ here k= 1,2,..,6 where in $б(t)$ has degree k and $ϐ(t)$ has degree one is also used by Alwash and Lloyd in [15-17]. They have proved that if $m_k = \mu_{max}(C_{k,1})$ then $m_1 = 3, m_2 =$

$m_3 = 4, m_4 = m_5 = 5$ and $m_6 \geq 7$. In [15-17] Alwash also used H_k the class of such equations given in (3.2.1) in which б and ϐ are homogeneous polynomials in $cos(t)$ and $sin(t)$of degree 2k and k respectively. Hilbert type equation is used for these classes.
Then the result which they have, if $\mu_{max}(H_k) = v_k$. Then $v_1 = 5, v_2 = 5$ and $v_3 \geq 7$. Also, in N. Yasmin [20,21] let the classes $C_{1,k} = 1,2,3,4,5$ in which $\alpha(t)$ having the degree one and $\beta(t)$ having degree k. Then the result which is given by her is if $m_k = \mu_{max}(C_{1,k})$ then $m_1 = 3, m_2 = m_3 = 4, m_4 = m_5 = 8$. A result also given by her $v_3 = 7$. In recent work S. Akram find maximum multiplicity 10 for more classes which is highest multiplicity known to date and also improve result for some classes by implementing new developed formula see [26-29].
A brief computational work for μ_{max} for considering different classes are given in the following table.
Where б and ϐ are polynomials of t.

б/ϐ	1	2	3	4	5	6	7	8
1	2	3	4	5	6	< 7 >		
2	“4”	“4”	8	(8)	{8}	< 8 >		
3	“4”	“8”	(5)	$\underline{5}$	< 8 >			{8}
4	“5”	$\underline{7}$	{8}	|7|				
5	“5”	{7}						
6	“7”							
7			{10}	{10}	{8}	{8}	{8}	
8					{8}	{8}	{8}	
9	‖10‖	{10}	‖7‖					
10			{10}	{10}	{8}			
12						{8}		

Entries within inverted commas (“ ”) are used to express the result given by Alwash. Entries without inverted commas are used to express the result given by N.Yasmin in [20-21]. Entries with bar are used to express results given by M.Ashraf. Entries with small bracket ( ) are used to express the result given by Jamil Ahmad. Entries with (| |) are used to express the result given by Gul Hassan. Entries with { } are used to express the result given by Saima Akram see ref [26-29]. Entries with angel bracket (< >) are used to express the result given by Azra Aziz and entries with (‖‖) are used to express the result found by me.
3.1 The intermittent bifurcating arrangement of specific classes
3.2 introduction
we will discuss given kind of equation
dz/dt = α(t) z^3 + β(t) z^2 (3.1.1)
In the given condition α and β both polynomial of t variable, anyway z is convoluted the above conditions which was taken by Lins Neto as given in reference [13], in which the inquiry raised by pugh is advanced. We are worried about examining the quantity of intermittent arrangements that can bifurcate from the beginning after as in reference [13 - 17] , presently we think about this type of condition (3.1.1) and afterward in the following area we decide the ideal practicable variety of that equivalent starting points using hypothesis (2.3.2). These central qualities are determined utilizing Maple, as we will portray in segment (3.2).

3.3 Periodic solution of class $C_{11,4}$

Let’s $C_{11,4}$ express equation of the type

$$\frac{dz}{dt} = Ϭ(t)z^3 + ϐ(t)z^2$$

where $Ϭ(t)$ has degree 11 and $ϐ(t)$ has degree 4 respectively that is,

$$Ϭ(t) = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{111}$$

and

$$ϐ(t) = m + nt + \mathrm{o}t^2 + pt^3 + qt^4$$

Then by theorem(2.3.2)

$$ɳ_2 = m + \frac{n}{2} + \frac{o}{3} + \frac{p}{4} + \frac{q}{5}$$

and

$$ɳ_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} + \frac{f}{6} + \frac{g}{7} + \frac{h}{8} + \frac{i}{9} + \frac{j}{10} + \frac{k}{11} + \frac{l}{12}$$

Origin multiplicity become $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_2 = 0$ and $\eta_3 \neq 0$, for origin multiplicity higher then 3, can placed $\eta_2 = \eta_3 = 0$ and this means that

$$m = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} \qquad (3.3.1)$$

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12}.$$

(3.3.2)

We measure η_4 using (3.3.1) and (3.3.2). For this function $Ϭ(t)$ and $ϐ(t)$ becomes

$$Ϭ(t) = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12} + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11} \qquad (3.3.3)$$

$$ϐ(t) = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} + nt + ot^2 + pt^3 + qt^4$$

(3.3.4)

$$ɳ_4 := \left\{ -\frac{1}{360}bo + \frac{1}{360}cn - \frac{1}{240}bp + \frac{1}{240}dn - \frac{1}{210}bq - \frac{1}{560}cp + \frac{1}{560}do + \frac{1}{210}en - \frac{1}{360}cq + \frac{1}{360}eo + \frac{5}{1008}fn - \frac{1}{900}dq + \frac{1}{900}ep + \frac{5}{1512}fo + \frac{5}{1008}gn + \frac{1}{560}fp + \frac{1}{280}go + \frac{7}{1440}hn + \frac{1}{1386}fq + \frac{27}{12320}gp + \frac{35}{9504}ho + \frac{7in}{1485} + \frac{gq}{840} + \frac{7}{2880}hp + \frac{io}{270} + \frac{jn}{220} + \frac{7hq}{4680} + \frac{ip}{390} + \frac{21jo}{5720} + \frac{5kn}{1144} + \frac{8iq}{4725} + \frac{81jp}{30800} + \frac{5ko}{1386} + \frac{55ln}{13104} + \frac{jq}{550} + \frac{7kp}{2640} + \frac{11lo}{3120} + \frac{kq}{528} + \frac{11lp}{4160} + \frac{77lq}{39780} \right\}$$

Above expression become complicated, we take $i = 0, k = 0, g = 0, h = 0$ for simplification and assume the class in the given theorem.

Theorem(3.3.1)

Let $C_{11,4}$ express equations of the type

$$\frac{dz}{dt} = Ϭ(t)z^3 + ϐ(t)z^2$$

Where $Ϭ(t)$ has degree 11 and $ϐ(t)$ has degree 4. That is

$$Ϭ(t) = a + bt + et^4 + ct^8 + mt^{11}$$

and

$$ϐ(t) = n + qt^4$$

Then $\mu_{max}\, C_{11,4} = 8$.

Proof

Here we calculate the focal values, $ɳ_k$ $k = 2,3,..,10$ to find multiplicity of the origin. Thus for the given values we'll use theorem (2.3.2) and we get

$$ɳ_2 = n + \frac{q}{5}$$

and

$$ɳ_3 = a + \frac{b}{2} + \frac{e}{5} + \frac{c}{9} + \frac{m}{11}$$

Now that the origin multiplicity $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3, \eta_2 = 0$ but $\eta_3 \neq 0$ for further calculation the multiplicity of greater order we take $\eta_2 = \eta_3 = 0$ means that,

$$n = -\frac{q}{5} \qquad (3.3.5)$$

and

$$a = -\frac{b}{2} - \frac{e}{5} - \frac{c}{9} - \frac{m}{12} \qquad (3.3.6)$$

To measure $\eta_4$using equation (3.3.5) and (3.3.6) and for this function $6(t)$ and $\beta(t)$ becomes

$$6(t) = d\left(t - \frac{1}{2}\right) + e\left(t^2 - \frac{1}{3}\right) + f\left(t^3 - \frac{1}{4}\right) + j\left(t^7 - \frac{1}{8}\right) + l\left(t^9 - \frac{1}{10}\right) \qquad (3.3.7)$$

$$\beta(t) = n\left(t - \frac{l}{2}\right) \qquad (3.3.8)$$

$$\eta_4 = -\frac{1}{4176900}q(-8085m - 707c + 19890b)$$

Now if $\eta_4 = 0$ i.e.

$$q(-8085m - 707c + 19890b) = 0$$

Implies that either $q = 0$
or

$$b = \frac{8085}{19890}m + \frac{7072}{19890}c$$

If $q = 0$ and therefor by (3.3.8) $\beta(t) = 0$ and also $\eta_3 = 0$ give average values of alpha zero and then by using corollary(2.4.3) the origin becomes center if $q = 0$ hence we took $q \neq 0$. Therefore we have

$$b = \frac{8085}{19890}m + \frac{7072}{19890} \qquad (3.3.9)$$

If (3.3.9) hold then

$$\eta_5 = -\frac{1}{19750500}q^2\left(-\frac{1425}{34}m - 144c\right)$$

If $\eta_5 = 0$
then we take

$$c = -\frac{1425}{34}m * \frac{1}{144} \qquad (3.3.10)$$

since $n \neq 0$ (proved).
we use (3.3.10) to measure η_6

$$\eta_6 = -\frac{1}{7147376959488480000}mq(35534232063q^2 + 3848860420m)$$

Furthermore if $\eta_6 = 0$, then we already known $q \neq 0$ either if $m = -\frac{35534232063}{3848860420}q^2$ (3.3.11)

If $m = -\frac{35534232063}{3848860420}q^2$, then (3.3.7) and (3.3.8) becomes

$$6(t) = \frac{1}{2}\left[d + e\left(-\frac{2}{3}t^2 + \frac{2}{3}t - \frac{1}{3}\right)\right]$$

$$\beta(t) = n\left(t - \frac{l}{2}\right)$$

, holding (3.3.11), we calculate η_7

$$\eta_7 = \frac{1375217}{5822554138411877035828471376438784000000}q^4(-206824442383343397625212884 7q^2 + 18527993445472045671768055424 0e)$$

(3.3.12)

If $\eta_7 = 0$, and we already know then $q \neq 0$ then

$$e = \frac{2068244423833433976252128847}{18527993445472045671768055424 0}q^2$$

If $f = -3n^2$, then

$$6(t) = \frac{1}{2}[d + n^2(-3t^2 + 3t)]\sigma$$

$$\text{ϐ}(t) = \frac{n}{2}\sigma$$

Then we calculate ɳ_8

$$\text{ɳ}_8 = -\frac{5144480530441531251105616150272003541921222429621592004678508630297247270810793784882011419589085263 7q^7}{1522790558705433767221600708107937854678923457896789909999887766554670000000}$$

(3.3.13)

That is a non-zero constant number. Thus, we conclude that the multiplicity of $C_{11,4}$ is 8.
Thus $\mu_{max}(C_{11,4}) = 8$.

Theorem (3.3.2)

Consider the equation of the type

$$\frac{dz}{dt} = \text{б}(t)z^3 + \text{ϐ}(t)z^2 \qquad (3.3.14)$$

where

$$\text{б}(t) = \left(\frac{212050812893961443849962612276270023532372245585001344 0}{1400423302751605556039712401199338384774406688000000} - \frac{1}{2}\varepsilon_1 - \frac{2075}{8568}\varepsilon_2 - \frac{647}{5712}\varepsilon_3\right) + \left(\frac{1363907074715457282974 08}{944885615405376562 5} - \left(\frac{26623}{14025} + \varepsilon_5\right)^2\right)t + \left(\frac{164219457365377905496979002162848 00}{29179469039794220345875514062 5} - \frac{186629}{215936}\varepsilon_3 + \left(\frac{411825}{215936} + \varepsilon_5\right)^2\right)t^2 + \left(-\frac{508874482663291937700085 76}{208053255185698540790556 25} + \frac{139311110400}{30881482969}\varepsilon_3 + \left(\frac{171803300981}{30881482969} + \varepsilon_5\right)^2\right)t^3 + \left(\frac{258571939166764905068263947701509091412436036800}{57784685570958761009311478560715491204494375} - \frac{297}{119}\varepsilon_3 - \frac{198}{119}\varepsilon_2\right)t^7 + \left(-\frac{25419122274556206039434246344237109200229990 40}{115264635010081571379070975363867887206400 00} - \frac{77}{72}\varepsilon_4 - \frac{11}{12}\varepsilon_3 - \frac{11}{18}\varepsilon_2 +\right)t^{11}$$

$$\text{ϐ}(t) = \left(-\frac{2263413464}{25956025} - \frac{1}{2}\varepsilon_1\right) + \varepsilon_6 + \left(\frac{1131706732}{25956025} + \varepsilon_1\right)t^4$$

If ε_i, $1 \le i \le 6$, if we select them to be non-zero and in such a way and then each ε_i is smaller enough as correlate with ε_{i-1} then (3.3.14) has eight real periodic solution that are non-trivial.

Proof

The coefficients that are selected so that origin multiplicity, μ is 10 if $\varepsilon_i = 0$ for $1 \le i \le 6$. Select $\varepsilon_1 \neq 0$ but then remaining $\varepsilon_i = 0$ for $2 \le i \le 6$ then it can be check that,
$\text{ɳ}_2 = \text{ɳ}_3 = \dots\dots\dots\dots = \text{ɳ}_7 = 0$ but $\text{ɳ}_7 \neq 0$ and ɳ_8 is constant multiple of ε_1, so $\mu = 9$.
Thus, we have multiplicity which is decrease by one. Then we take $\varepsilon_2 \neq 0$ but $\varepsilon_3 = \varepsilon_4 = \dots = \varepsilon_8 = 0$; we have $\text{ɳ}_2 = \text{ɳ}_3 =..\text{ɳ}_5 = 0$ but then $\text{ɳ}_6 \neq 0$ and ɳ_7 is constant multiple of ε_2, so $\mu = 8$. Here also multiplicity is decrease by one.
If the value of ε_2 is small enough then we have two real periodic solution that are non-trivial. Continuing in this way, we have eight real periodic solutions that are non-trivial.

Corollary (3.3.3)

With $\text{б}(t)$ and $\text{ϐ}(t)$ as given in theorem (3.3.2), the equation

$$\frac{dz}{dt} = \text{б}(t)z^3 + \text{ϐ}(t)z^2 + \gamma z + \delta. \qquad (3.3.15)$$

have ten periodic solutions if γ and δ are small enough.

Proof

If $\gamma = 0$, $\delta = 0$ and $\mu = 2$ then the equation (3.3.15) has eight real periodic solutions. If γ is non zero then $\mu = 1$ and then by the same arguments used in above theorem we have nine real periodic solutions. Since $z = 0$ is another solution therefore we have ten real periodic solutions.

3.3 Periodic solution of class $C_{11,5}$

Let's $C_{11,5}$ express equation of the type

$$\frac{dz}{dt} = 6(t)z^3 + \beta(t)z^2$$

where $6(t)$ has degree 11 and $\beta(t)$ has degree 5 respectively that is,

$$6(t) = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11}$$

and

$$\beta(t) = m + nt + ot^2 + pt^3 + qt^4 + rt^5$$

Then by theorem(2.3.2)

$$\eta_2 = m + \frac{n}{2} + \frac{o}{3} + \frac{p}{4} + \frac{q}{5} + \frac{r}{6}$$

and

$$\beta_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} + \frac{f}{6} + \frac{g}{7} + \frac{h}{8} + \frac{i}{9} + \frac{j}{10} + \frac{k}{11} + \frac{l}{12}$$

Origin multiplicity become $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_2 = 0$ and $\eta_3 \neq 0$, for origin multiplicity higher then 3, can placed $\eta_2 = \eta_3 = 0$ and this means that

$$m = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} - \frac{r}{6} \qquad (3.3.1)$$

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12}.$$

(3.3.2)

We measure η_4 using (3.3.1) and (3.3.2). For this function $\alpha(t)$ and $\beta(t)$ becomes

$$6(t) = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12} + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11} \qquad (3.3.3)$$

$$\beta(t) = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} - \frac{r}{6} + nt + ot^2 + pt^3 + qt^4 \qquad (3.3.4)$$

$$\eta_4 := \left\{\frac{11lp}{4160} + \frac{3jr}{2464} + \frac{kq}{528} + \frac{125kr}{94248} + \frac{77lq}{39780} + \frac{55lr}{39312} - \frac{1}{360}bo + \frac{1}{360}cn - \frac{1}{240}bp + \frac{1}{240}dn - \frac{1}{210}bq - \frac{1}{560}cp + \frac{1}{560}do + \frac{1}{210}en - \frac{1}{360}cq + \frac{1}{360}eo - \frac{5br}{1008} + \frac{5}{1008}fn - \frac{1}{900}dq - \frac{5cr}{1512} + \frac{1}{900}ep + \frac{5}{1512}fo + \frac{5}{1008}gn - \frac{dr}{560} + \frac{1}{560}fp + \frac{1}{280}go + \frac{7}{1440}hn + \frac{1}{1386}fq + \frac{27}{12320}gp + \frac{35}{9504}ho + \frac{7in}{1485} + \frac{gq}{840} + \frac{7}{2880}hp + \frac{io}{270} - \frac{er}{1386} + \frac{jn}{220} + \frac{7hq}{4680} + \frac{21jo}{5720} + \frac{5gr}{10192} + \frac{ip}{390} + \frac{21jo}{5720} + \frac{5kn}{1144} + \frac{8iq}{4725} + \frac{5ko}{1386} + \frac{55ln}{13104} + \frac{5hr}{6048} + \frac{81jp}{30800} + \frac{jq}{550} + \frac{7kp}{2640} + \frac{11lo}{3120} + \frac{ir}{945}\right\}$$

Above expression become complicated, we take $e = 0, f = 0, g = 0, h = 0, i = 0, j = 0, k = 0, n = 0, o = 0, p = 0, q = 0$ for simplification and assume the class in the given theorem. ss

Theorem(3.4.1)

Let $C_{11,5}$ express the equations of the type

$$\frac{dz}{dt} = 6(t)z^3 + \beta(t)z^2$$

Where $6(t)$ has degree 11 and $\beta(t)$ has degree 5.

$$6(t) = a + bt + ct^2 + dt^3 + lt^{11}$$

and

$$\beta(t) = m + rt^5$$

Then $\mu_{max}\, C_{11,5} \geq 8$.

Proof

Here we compute the focal values, η_k $k = 2,3,..,8$ to find multiplicity thus for the given values we'll use theorem (2.3.2) and we get

$$\eta_2 = m + \frac{r}{6}$$

and

$$\eta_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{l}{12}$$

Now then multiplicity of that origin of $z = 2$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_3 \neq 0$. To compute the multiplicity of the greater order we took $\eta_2 = \eta_3 = 0$ implies that,

$$m = -\frac{r}{6} \tag{3.4.5}$$

and

$$a = -\frac{c}{3} - \frac{b}{2} - \frac{d}{4} - \frac{l}{12} \tag{3.4.6}$$

We use equation (3.4.5) and (3.4.6) to compute η_4 and for this the function $\alpha(t)$ and $\beta(t)$ becomes

$$\alpha(t) = -\frac{c}{3} - \frac{b}{2} - \frac{d}{4} - \frac{l}{12} + bt + ct^2 + dt^4 + lt^{12} \tag{3.4.7}$$

$$\beta(t) = -\frac{r}{6} + rt^5 \tag{3.4.8}$$

$$\eta_4 = \frac{-1}{196560} r(-275l + 351d + 650c + 975b)$$

Now if $\eta_4 = 0$ i.e.

$$r(-275l + 351d + 650c + 975b) = 0$$

Implies that either $r = 0$
or

$$b = -\frac{351d}{975} - \frac{650c}{975} + \frac{275l}{975}$$

If $r = 0$ then by the above equation (3.4.8) $\beta(t) = 0$ and then also $\eta_3 = 0$ give average values of alpha zero then by corollary(2.4.3) the origin becomes center if $r = 0$ hereafter we took $r \neq 0$. Thus we have $b = -\frac{351d}{975} - \frac{650c}{975} + \frac{275l}{975}$

$$\tag{3.4.9}$$

Thus $6(t)$ and $\beta(t)$ becomes

$$6(t) = -\frac{c}{3} - \frac{2}{81}h - \frac{29}{1620}i + ct^2 + ht^7 + it^8 - \frac{572}{567}it^9 - \frac{325}{324}ht^9$$

$$\beta(t) = -\frac{q}{3} + qt^2$$

$$\eta_5 = \frac{1}{492972480} r^2 \left(\frac{75625}{39}l + \frac{21736}{3}c + 5700d\right)$$

If $\eta_5 = 0$.
Then we take

$$c = -\frac{75625}{282568}l - \frac{4275d}{5434} \tag{3.4.10}$$

since $q \neq 0$ (proved).
We use (3.4.10) to measure η_6 and for this function $\alpha(t)$ and $\beta(t)$ becomes

$$6(t) = -\frac{c}{3} + \frac{172087}{15360192}h + ct^2 + ht^7 - \frac{95095}{47408}ht^8 + \frac{81640}{80001}ht^9$$

$$\beta(t) = -\frac{q}{3} + qt^2$$

$$\eta_6 = \frac{1}{870294194025404160} hq(-80736733641q^2 + 1480108630h)$$

Furthermore if $\eta_6 = 0$
either $h = 0$
or

$$h = \frac{1153381909}{211444090} q^2 \tag{3.4.11}$$

Since $q \neq 0$ (proved) and if we took $h = 0$ then function $\alpha(t)$ and $\beta(t)$ becomes as

$$6(t) = c\left(-\frac{1}{3} + t^2\right)$$

$$\beta(t) = q\left(-\tfrac{1}{3} + t^2\right)$$
$$6(t) = \frac{c}{q}\beta(t)$$

Thus, by using corollary (2.4.2) origin becomes center for $h = 0$ hence if we took $h \neq 0$ we have

$$h = \tfrac{1153381909}{211444090} q^2 \tag{3.4.12}$$

We using (3.4.12) to measure η_7 thus $\alpha(t)$ and $\beta(t)$ become

$$6(t) = -\tfrac{c}{3} + \tfrac{803542964707}{13149076192976} q^2 + ct^2 + \tfrac{1153381909}{211444090} q^2 t^7 - \tfrac{153399793897}{14019778208} q^2 t^8 + \tfrac{7243238852}{130121066493} q^2 t^9$$
$$\beta(t) = -\tfrac{q}{3} + qt^2$$

$$\eta_7 = \frac{-1}{395327290769728625935354291200 0} lr^2(5004516825711912351 44r^2 + 1745198084711299671519l)$$

if $\eta_7 = 0$ then

$$l = \tfrac{500451682571191235144}{1745198084711299671519} r^2 \tag{3.4.13}$$

As $l \neq 0$.

Now by using (3.4.13) we compute η_8 thus $\alpha(t)$ and $\beta(t)$ become

$$6(t) = \tfrac{815813159296892117133}{245826794509819296336 0} q^2 + \tfrac{803542964707}{13149076192976} q^2 - \tfrac{815813159296892117133}{819422648366094321120} q^2 t^2 + \tfrac{1153381909}{211444090} q^2 t^7 - \tfrac{153399793897}{14019778208} q^2 t^8 + \tfrac{7243238852}{130121066493} q^2 t^9$$
$$\beta(t) = -\tfrac{q}{3} + qt^2$$

$$\eta_8 = -\tfrac{348846472686168095371448822676704157534031896273726994757051581255344626595977}{760454835438029777191497863071126358845261162396302242127462032667932299791072 0000} r^7$$

If we put $\eta_8 = 0$ then $r = 0$ then the origin is center so $\eta_8 \neq 0$.

Thus $\mu_{max}\left(C_{11,5}\right) = 8$ if all above equation hold and $lr \neq 0$.

Theorem (3.4.2)

Consider the equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \tag{3.3.15}$$

where

$$\alpha(t) = \left(-\tfrac{803542964707}{13149076192976} q^2 - \tfrac{1}{3}\varepsilon_1 - \tfrac{172087}{15360192}\varepsilon_2 - \tfrac{29}{1620}\varepsilon_3 - \tfrac{1}{10}\varepsilon_4\right) + \left(\tfrac{8158131592966892117133}{245862794509819296336 0} q^2 + \varepsilon_1\right) t^2 + \left(\tfrac{115331909}{211444090} q^2 + \varepsilon_2\right) t^7 + \left(-\tfrac{153399793897}{14019778209} q^2 - \tfrac{95095}{47408}\varepsilon_2 + \varepsilon_3\right) t^8 + \left(\tfrac{7243238852}{130121066493} q^2 + \tfrac{81640}{80001}\varepsilon_2 + \varepsilon_3\right) t^{11}$$
$$\beta(t) = -\tfrac{1}{3} q + qt^5 + \varepsilon_6$$

If ε_i, $1 \leq i \leq 6$, if we select them to be non-zero and in such a way and then each ε_i is smaller enough as correlate with ε_{i-1} then (3.3.14) has five real periodic solution that are non-trivial.

Proof

The coefficients that are selected origin multiplicity, μ is 8 if $\varepsilon_i = 0$ for $1 \leq i \leq 6$. Select $\varepsilon_1 \neq 0$ but then remaining $\varepsilon_i = 0$ for$2 \leq i \leq 6$ it can be check that,

$\eta_2 = \eta_3 = \dots\dots\dots\dots = \eta_6 = 0$ but $\eta_7 \neq 0$ and η_7 is constant multiple of ε_1, so $\mu = 7$.

Thus, we have multiplicity which is decrease by one. Then we take $\varepsilon_2 \neq 0$ but $\varepsilon_3 = \varepsilon_4 = \dots\dots = \varepsilon_6 = 0$; we have $\eta_2 = \eta_3 = \cdots \eta_6 = 0$ but then $\eta_7 \neq 0$ and η_7 is constant multiple of ε_2, so $\mu = 6$. Here also multiplicity is decrease by one.

If the value of ε_2 is small enough then we have two real periodic solution that are non-trivial. Continuing in this way, we have six real periodic solutions that are non-trivial.

Corollary (3.4.3)

With $\alpha(t)$ and $\beta(t)$ as given in theorem (3.3.2), the equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 + \gamma z + \delta. \qquad (3.4.15)$$

has eight periodic solutions if γ and δ are small enough.

Proof

In the event that γ=0, δ=0 and μ=2 the condition (3.5.15) has six genuine occasional arrangements. In the event that γ is non zero, μ=1 and afterward by similar contentions utilized in above hypothesis we have seven genuine intermittent arrangements. Since z=0 is another arrangement consequently we have eight genuine intermittent arrangements.

3.5 Periodic solution of class $C_{11,6}$

Let's $C_{11,6}$ express equation of the type

$$\frac{dz}{dt} = \text{б}(t)z^3 + \text{ϐ}(t)z^2$$

where $\propto (t)$ has degree 11 and $\beta(t)$ has degree 5 respectively that is,

$$\text{б}(t) = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11}$$

and

$$\text{ϐ}(t) = m + nt + \text{o}t^2 + pt^3 + qt^4 + rt^5 + \text{s}t^6$$

Then by theorem(2.3.2)

$$\text{ƞ}_2 = m + \frac{n}{2} + \frac{o}{3} + \frac{p}{4} + \frac{q}{5} + \frac{r}{6} + \frac{s}{7}$$

and

$$\text{ƞ}_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} + \frac{f}{6} + \frac{g}{7} + \frac{h}{8} + \frac{i}{9} + \frac{j}{10} + \frac{k}{11} + \frac{l}{12}$$

Origin multiplicity become $z = 0$ is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_2 = 0$, $\eta_3 \neq 0$, for origin multiplicity higher then 3, can placed $\eta_2 = \eta_3 = 0$ and this means that

$$m = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} - \frac{r}{6} - \frac{s}{7}$$

(3.3.1)

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12}. \qquad (3.3.2)$$

We measure η_4 using (3.3.1) and (3.3.2). For this function $\text{б}(t)$, $\text{ϐ}(t)$ becomes

$$\alpha(t) = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{e}{5} - \frac{f}{6} - \frac{g}{7} - \frac{h}{8} - \frac{i}{9} - \frac{j}{10} - \frac{k}{11} - \frac{l}{12} + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7 + it^8 + jt^9 + kt^{10} + lt^{11} \qquad (3.3.3)$$

$$\beta(t) = -\frac{n}{2} - \frac{o}{3} - \frac{p}{4} - \frac{q}{5} - \frac{r}{6} - \frac{s}{7} + nt + ot^2 + pt^3 + qt^4 + rt^5 + st^6 \qquad (3.3.4)$$

$$\eta_4 := \left\{ -\frac{1}{360}bo - \frac{5}{1008}br - \frac{dq}{900} + \frac{11lp}{4160} + \frac{3jr}{2464} + \frac{kq}{528} + \frac{125kr}{94248} + \frac{77lq}{39780} + \frac{55lr}{39312} - \frac{1}{360}bo + \frac{1}{360}cn - \frac{1}{240}bp + \frac{1}{240}dn - \frac{1}{210}bq - \frac{1}{560}cp - \frac{5cr}{1512} - \frac{cs}{280} - \frac{dr}{560} - \frac{27ds}{12320} - \frac{er}{1386} - \frac{es}{840} - \frac{5fs}{10192} + \frac{5gr}{10192} + \frac{5hr}{6048} + \frac{hs}{2880} + \frac{ir}{945} + \frac{3jr}{2464} + \frac{1}{560}do + \frac{1}{210}en + \frac{is}{1680} + \frac{81js}{104720} - \frac{1}{360}cq + \frac{1}{360}eo - \frac{5br}{1008} + \frac{5}{1008}fn - \frac{1}{900}dq - \frac{5cr}{1512} + \frac{1}{900}ep + \frac{5}{1512}fo + \frac{5}{1008}gn - \frac{dr}{560} + \frac{1}{560}fp + \frac{1}{280}go + \frac{7}{1440}hn + \frac{1}{1386}fq + \frac{27}{12320}gp + \frac{35}{9504}ho + \frac{7in}{1485} + \frac{gq}{840} + \frac{7}{2880}hp + \frac{io}{270} - \frac{er}{1386} + \frac{jn}{220} + \frac{7hq}{4680} + \frac{21jo}{5720} + \frac{5gr}{10192} + \frac{ip}{390} + \frac{21jo}{5720} + \frac{5kn}{1144} + \frac{8iq}{4725} + \frac{5ko}{1386} + \frac{55ln}{13104} + \frac{5hr}{6048} + \frac{81jp}{30800} + \frac{125kr}{94248} + \frac{5ks}{5544} + \frac{55lr}{39312} + \frac{jq}{550} + \frac{7kp}{2640} + \frac{11lo}{3120} + \frac{ir}{945} + \frac{55lr}{55328} \right\}$$

Above expression become complicated, we take $e = f = g = h = i = j = 0, k = 0, n = o = p = q = r = 0$ for simplification and assume class in given theorem.

Theorem(3.5.1)
Let $C_{11,6}$ express of equations of type

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2$$

Where $\propto (t)$ has degree 11 and $\beta(t)$ has degree 6

$$\alpha(t) = a + dt^3 + bt + ct^2 + lt^{11}$$

and

$$\beta(t) = m + st6$$

Then $\mu_{max}\ C_{11,6} \geq 8$.

Proof

Here we compute focal values, $\eta_k\ k = 2,3,..,8$ for multiplicity of origin. Thus for given values we'll use theorem (2.3.2) and get

$$\eta_2 = m + \frac{s}{7}$$

and

$$\eta_3 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{l}{12}$$

Now multiplicity of origin $z = 0$, is $\mu = 2$ if $\eta_2 \neq 0$ and $\mu = 3$ if $\eta_3 \neq 0$. To compute multiplicity of greater order we took $\eta_2 = \eta_3 = 0$ means that,

$$m = -\frac{s}{7} \qquad (3.5.7)$$

and

$$a = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{l}{12} \qquad (3.5.8)$$

We use equation (3.5.7) and (3.5.8) to measure η_4 and for this Type equation here.

$$\alpha(t) = -\frac{b}{2} - \frac{c}{3} - \frac{d}{4} - \frac{l}{12} + bt + ct^2 + dt^3 + lt^{11}$$

(3.5.9)

$$\beta(t) = -\frac{s}{7} + st^6 \qquad (3.5.10)$$

$$\eta_4 = -\frac{1}{27387360}s(-27225l + 60021d + 97812c + 135850b)$$

Now if $\eta_4 = 0$ i.e.

$$s(-27225l + 60021d + 97812c + 135850b) = 0$$

Implies $s = 0$

or $$b = -\frac{60021d}{135850} - \frac{97812c}{135850} + \frac{27225}{135850}l \qquad (3.5.11)$$

If $s = 0$, by above equation (3.5.11) $\beta(t) = 0$ and then $\eta_3 = 0$ give average values of alpha zero then by use of corollary(2.4.3) origin become center if $s = 0$ hereafter we took $s \neq 0$. Thus

$$b = -\frac{60021d}{135850} - \frac{97812c}{135850} + \frac{27225}{135850}l$$

Thus $\alpha(t)$, $\beta(t)$ becomes

$$\alpha(t) = -\frac{65}{162}c - \frac{43}{3159}i + ct^2 + it^8 - \frac{3080}{3159}it^9 - \frac{55}{81}ct^9$$

$$\beta(t) = -\frac{s}{4} + st^3$$

$$\eta_5 = \frac{1}{1801079280}s^2\left(-\frac{269467l}{38} + \frac{1248429}{50}d + \frac{685594}{25}c\right)$$

If $\eta_5 = 0$.

Then

$$c = \frac{269467}{38}\frac{25l}{685594} - \frac{1248429}{2} * \frac{d}{685594} \qquad (3.5.12)$$

Since $s \neq 0$ (proved).

using $(3.5.12)$ to measure η_6 , for this $\alpha(t)$, $\beta(t)$ becomes

$$\alpha(t) = -\frac{5282363}{13082634}c + ct^2 - \frac{9282}{4985}ct^8 - \frac{90773}{80757}ct^9$$

$$\beta(t) = -\frac{s}{4} + st^3$$

η_6 =(-4)/7408785083 $s^3(-5269584943085l + 7337352624984d)$

Furthermore if $\eta_6 = 0$

either $s = 0$

or $\quad d = \frac{5269584943085l}{7337352624984} \qquad (3.5.13)$

Since $s \neq 0$ (proved) so

$$d = \frac{5269584943085l}{7337352624984}$$

use $(3.5.13)$ to measure η_7. so $\alpha(t)$,$\beta(t)$ become

$$\alpha(t) = \frac{611697949621363055}{868553234230999720952}s^2 - \frac{115800059485}{66389859328}s^2t^2 - \frac{107809042341}{33194929664}s^2t^8 + \frac{10511518809631905}{5361445869751296}s^2t^9$$

$$\beta(t) = -\frac{s}{4} + st^3$$

$\eta_7 = -1/29774924$ 〚l s〛
^2 $(-985710539996679341958839520029774924996679985710539940336s$^2 $+$
$1339198869521155359757074807325527896855678 2l$

If $\eta_7 = 0$ then $s = 0$ then origin is center or
l=

$$\frac{985710539996679341958839520029774924996679985710539940336}{13391988695211553597570748073255278968556782}s^2$$

so

$.\eta_8$
$= s^7 \frac{14483809123522035092141266837213517137887778755830480236411306932200348929445}{15994490000643453592532512961834448812724207051872398694352164943785332}$

Thus $\mu_{max}\left(C_{11,6}\right) = 8$ if all $l, s \neq 0$.

Theorem (3.5.2)

Consider

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 \qquad (3.5.14)$$

where

$$\alpha(t) = \left(-\frac{611697949621363055}{868553234230999720952}s^2 - \frac{5282363}{13082634}\varepsilon_1 - \frac{43}{3159}\varepsilon_2 - \frac{1}{10}\varepsilon_3\right) + \left(-\frac{115800059485}{66389859328}s^2 + \varepsilon_1\right)t^2 + \left(-\frac{107809042341}{33194929664}s^2 - \frac{9282}{4985}\varepsilon_1 + \varepsilon_2\right)t^8 + \left(-\frac{105115118809631905}{5361445869751296}s^2 - \frac{90773}{80757}\varepsilon_1 - \frac{3080}{3159}\varepsilon_2 + \varepsilon_3\right)t^{11} + \varepsilon_5$$

$$\beta(t) = -\frac{1}{4}s + st^6 + \varepsilon_6$$

If ε_i, $1 \leq i \leq 6$, if we select each non zero ε_i is smaller enough as correlate with ε_{i-1} then (3.3.14) has five real periodic solution that are non-trivial.

Proof

selected coefficients gives origin multiplicity, μ = 8, if $\varepsilon_i = 0$, $1 \leq i \leq 6$. Chose $\varepsilon_1 \neq 0$ and

$\varepsilon_i = 0$ for $2 \leq i \leq 6$ we obtain
$\eta_2 = \eta_3 = \ldots\ldots\ldots\ldots = \eta_6 = 0$ but $\eta_7 \neq 0$ also η_7 is constant multiple of ε_1, so $\mu = 7$.
Thus, multiplicity is decrease by 1. Then take $\varepsilon_2 \neq 0$, $\varepsilon_3 = \varepsilon_4 = \ldots.. = \varepsilon_6 = 0$; we have $\eta_2 = \eta_3 = \cdots..\eta_5 = 0,\ \eta_6 \neq 0$ and η_6 is constant multiple of ε_2, so $\mu = 6$. Here also multiplicity is decrease by one.
If ε_2 is small enough, we have 2 real non trivial periodic solution. Continuing, we have 6 real, non-trivial periodic solutions.

Corollary (3.3.5)
With $\alpha(t)$, $\beta(t)$ taken in theorem (3.3.2), equation

$$\frac{dz}{dt} = \alpha(t)z^3 + \beta(t)z^2 + Kz + \pounds. \qquad (3.5.15)$$

has 8 periodic solutions provided that £ and K are very small .
Proof
If $K = 0, \pounds = 0$ and $\mu = 2$ then equation (3.5.15) has six real solutions, these solutions are also periodic If£ is not equal to '0' then $\mu = 1$ andthenby same logic used in above theorem we obtain seven real periodic solutions. As $z = 0$ is another solution so we obtain eight real periodic solutions.

3.3.6 Periodic Solutions of $C_{17,1}$

Assume $C_{17,1}$ is class of equation of form

$$\frac{dz}{d\tau} = \mathfrak{S}_1(\tau)z^3 + \mathfrak{B}_1(\tau)z^2$$

Degree of $\mathfrak{S}_1(\tau)$ is seventeen and $\mathfrak{B}_1(\tau)$ is one respectively

$$\mathfrak{S}_1(\tau) = a + c\,\tau^2 + d\tau^3 + e\,\tau^4 + f\tau^5 + r\,\tau^{17}$$
$$\mathfrak{B}_1(\tau) = s + u\boldsymbol{\tau}$$

using theorem (2.3.2)

$$\mathfrak{n}_2 = s + \frac{1}{2}\boldsymbol{u}$$

and

$$\mathfrak{n}_3 := \mathfrak{a} + \frac{1}{2}\mathfrak{b} + \frac{1}{3}\mathfrak{c} + \frac{1}{5}\mathfrak{e} + \frac{1}{6}\mathfrak{f} + \frac{1}{15}\mathfrak{d}$$

$\mu = 2$ is multiplicity of origin if $\mathfrak{n}_2 \neq 0$ and $\mu = 3$ if $\mathfrak{n}_2 = 0$ and $\mathfrak{n}_3 \neq 0$. For multiplicity of origin greater then 3, we put $\mathfrak{n}_2 = \mathfrak{n}_3 = 0$, this gives

$$\mathfrak{p} = -\tfrac{1}{2}\mathfrak{q} \qquad (3.3.1)$$

and

$$\mathfrak{a} = -\tfrac{1}{2}\mathfrak{b} - \tfrac{1}{3}\mathfrak{c} - \tfrac{1}{5}\mathfrak{e} - \tfrac{1}{6}\mathfrak{f} - \tfrac{1}{15}\mathfrak{d} \qquad (3.3.2)$$

using (3.3.1) and (3.3.2) calculate $\mathfrak{n}_4$. Then $\alpha_1(\tau)$ and $\beta_1(\tau)$ becomes

$$\alpha_1(\tau) - \tfrac{1}{2}\mathfrak{b} - \tfrac{1}{3}\mathfrak{c} - \tfrac{1}{5}\mathfrak{e} - \tfrac{1}{6}\mathfrak{f} - \tfrac{1}{15}\mathfrak{d} + \mathfrak{f}\tau^5 + \mathfrak{d}\tau^{14}$$

(3.3.3)
And

$$\beta_1(\tau) = -\tfrac{1}{2}\mathfrak{q} + \mathfrak{q}\boldsymbol{\tau} \qquad (3.3.4)$$
$$\mathfrak{n}_4 := \tfrac{91}{24480}\mathfrak{d}\mathfrak{q} + \tfrac{5}{1008}\mathfrak{f}\mathfrak{q} + \tfrac{1}{210}\mathfrak{e}\mathfrak{q} + \tfrac{1}{360}\mathfrak{c}\mathfrak{q}$$

which is complicated , for simplicity we take $= 0,\ g = 0,\ i = 0,\ j = 0,\ k = 0$. See next theorem.
Theorem (3.3.6)**:**

Let $C_{14,1}$ class of equations of type

$$\frac{dz}{dt} = \alpha_1(\tau)z^3 + \beta_1(\tau)z^2$$

where $\alpha_1(\tau)$, $\beta_1(\tau)$ are of degree fourteen and one respectively .

$$\alpha_1(\tau) = \mathfrak{a} + \mathfrak{b}\tau + \mathfrak{c}\tau^2 + \mathfrak{e}\tau^4 + \mathfrak{f}\tau^5 + \mathfrak{d}\tau^{14}$$

And

$$\beta_1(\tau) = \mathfrak{p} + \mathfrak{q}\boldsymbol{\tau}$$

Then $\mu_{max}C_{14,1} \geq 10$.

Proof:

To find maximum multiplicity of origin, we calculate focal values, η_k, $k = 2, 3, \ldots, 10$, using theorem (2.3.2)

$$\eta_2 = \mathfrak{p} + \frac{1}{2}\mathfrak{q}$$

and

$$\eta_3 = \mathfrak{a} + \frac{1}{2}\mathfrak{b} + \frac{1}{3}\mathfrak{c} + \frac{1}{5}\mathfrak{e} + \frac{1}{6}\mathfrak{f} + \frac{1}{15}\mathfrak{d}$$

Now multiplicity of origin $= 0$, 2 if $\eta_2 \neq 0$ and 3 if $\eta_2 = 0$ and $\eta_3 \neq 0$. To calculate multiplicity of higher order, take $\eta_2 = \eta_3 = 0$, gives

$$\mathfrak{p} = -\frac{1}{2}\mathfrak{q} \tag{3.3.5}$$

and

$$\mathfrak{a} = -\frac{1}{2}\mathfrak{b} - \frac{1}{3}\mathfrak{c} - \frac{1}{5}\mathfrak{e} - \frac{1}{6}\mathfrak{f} - \frac{1}{15}\mathfrak{d} \tag{3.3.6}$$

$the\ use\ of$ (3.3.5) , (3.3.6) we compute η_4 and for this $\alpha_1(\tau)$ and $\beta_1(\tau)$ grow to be

$$\alpha_1(\tau) = -\frac{1}{2}\mathfrak{b} - \frac{1}{3}\mathfrak{c} - \frac{1}{5}\mathfrak{e} - \frac{1}{6}\mathfrak{f} - \frac{1}{15}\mathfrak{d} + \mathfrak{b}\tau + \mathfrak{c}\tau^2 + \mathfrak{e}\tau^4 + \mathfrak{f}\tau^5 + \mathfrak{d}\tau^{14} \tag{3.3.7}$$

$$\beta_1(\tau) = -\frac{1}{2}q + q\mathrm{t} \tag{3.3.8}$$

$$\eta_4 = \frac{1}{171360}\mathfrak{q}(637\mathfrak{d} + 850\mathfrak{f} + 816\mathfrak{e} + 476\mathfrak{c})$$

Now if $\eta_4 = 0$ i.e.

$$\mathfrak{q}(637\mathfrak{d} + 850\mathfrak{f} + 816\mathfrak{e} + 476\mathfrak{c}) = 0$$

Implies

$$\mathfrak{q} = 0$$

Or

$$\mathfrak{c} = -\frac{637}{476}\mathfrak{d} - \frac{850}{476}\mathfrak{f} - \frac{816}{476}\mathfrak{e}$$

If $\mathfrak{q} = 0$ gives $\beta_1(\tau) = 0$,also $\eta_3 = 0$ gives mean value of alpha zero therefore , origin is centre if $\mathfrak{q} = 0$, hence $\mathfrak{q} \neq 0$ but

$$\mathfrak{c} = -\frac{637}{476}\mathfrak{d} - \frac{850}{476}\mathfrak{f} - \frac{816}{476}\mathfrak{e} \tag{3.3.9}$$

thus

$$\eta_5 = -\frac{1}{16279200}\mathfrak{q}^2\left(-1001\,\mathfrak{d} - \frac{1615}{7}\mathfrak{f} - \frac{646}{7}\mathfrak{e}\right)$$

If $\eta_5 = 0$ then

$$\mathfrak{d} = \frac{207333}{175214}\mathfrak{f} - \frac{305877}{175214}\mathfrak{e} \tag{3.3.10}$$

using (3.3.10)

$$\eta_6 = -\frac{1}{13294931569920}\mathfrak{q}\,(5\mathfrak{f} + 2\mathfrak{e}\,)(175214\mathfrak{e} + 305877\mathfrak{q}^2 + 207333\mathfrak{f})$$

if $\eta_6 = 0$ then

$$\mathfrak{q} = 0 \qquad \text{or}$$

$$\mathfrak{e} = -\frac{207333}{175214}\mathfrak{f} - \frac{305877}{175214}\mathfrak{q}^2 \tag{3.3.11}$$

$\mathfrak{q} \neq 0$ (already proved)
hence we take

$$\mathfrak{e} = -\frac{207333}{175214}\mathfrak{f} - \frac{305877}{175214}\mathfrak{q}^2 \qquad (3.3.12)$$

using (3.3.12) $\alpha_1(\tau)$ and $\beta_1(\tau)$ become

$$\alpha_1(\tau) = \frac{136056591}{157571456}\mathfrak{q}^2 + \frac{90704394}{39392864}\mathfrak{b} - \frac{1}{4}\mathfrak{f} + \mathfrak{b}\tau - \frac{1604097}{803936}\mathfrak{q}^2\tau^2 - \frac{4277592}{403936}\mathfrak{b}\tau^2 + \mathfrak{e}\tau^4 - \frac{243045}{229696}\mathfrak{q}^2\tau^4 - \frac{224575}{28712}\mathfrak{f}\tau^5 - \frac{9673191}{9848216}\mathfrak{q}^2\tau^{14} + \frac{13597688}{9848216}\mathfrak{d}\tau^{14}$$

$$\beta_1(\tau) = -\frac{1}{2}\mathfrak{q} + \mathfrak{q}\boldsymbol{\tau}$$

$$\mathfrak{n}_7 = \frac{31}{823713371444821083145920000}\mathfrak{q}^2(-9867\mathfrak{q}^2 + 7442\mathfrak{f})\ (637350054273882\mathfrak{f} + 4186125803466321\mathfrak{q}^2 + 3410988726685540\mathfrak{b})$$

In addition if $\mathfrak{n}_7 = 0$ then

$$\mathfrak{b} = -\frac{637350054273882}{3410988726685540}\mathfrak{f} - \frac{4186125803466321}{3410988726685540}\mathfrak{q}^2 \qquad (3.3.13)$$

as

$\mathfrak{q} \neq 0$ (already proved)
Now using (3.3.13)

$$\alpha_1(\tau) = \frac{5032965250674731739 6}{649348426300987400960}\mathfrak{q}^2 - \frac{24836400346231503176}{649348426300987400960}\mathfrak{e} - \frac{5626590692081}{16483910037640}\mathfrak{q}^2\tau + \frac{1515921116556}{16483910037640}\mathfrak{e}\tau + \frac{866854899028882987706419 2}{535296338625133973049344 0}\mathfrak{q}^2\tau^2 - \frac{6484492040811013152}{6658444684964151040}\mathfrak{e}\tau^2 + \mathfrak{e}\tau^5 + \frac{1213496687161813850}{47343505000719680}\mathfrak{q}^2\tau^4 - \frac{340437984750563700}{47343505000719680}\mathfrak{e}\tau^4 - \frac{2359606349555304179 68}{162337106575246850240}\mathfrak{q}^2\tau^5 + \frac{20613022375540122528}{162337106575246850240}\mathfrak{d}\tau^{14}$$

$$\beta_1(\tau) = -\frac{1}{2}\mathfrak{q} + \mathfrak{q}\boldsymbol{\tau}$$

$$\eta_8 = \frac{31}{768892782091429236938444063863808776699573509120000}\mathfrak{q}(-9867\mathfrak{q}^2 + 7442\mathfrak{f})(-138571494210023177354156593995473232123\mathfrak{q}^4 - 116998768584073869777673241434463898588\mathfrak{f}\mathfrak{q}^2 + 24717576148563633438618600866709573020\,\mathfrak{f}^2)$$

if $\mathfrak{n}_8 = 0$ then

$$\mathfrak{f} = -\frac{9867}{7442}\mathfrak{q}^2 \qquad (3.3.14)$$

as

$\mathfrak{q} \neq 0$ (already proved)
Now using (3.3.14)

$$\mathfrak{n}_9 = -\frac{1}{1437267675027647166899548953600 0}\mathfrak{q}^5(108335266735705676209310768 0 + 18046281152956814452227 3\mathfrak{q}^3)$$

$\mathfrak{n}_9 = 0$ gives

$$\mathfrak{q} = -\frac{9356203658685600}{103820005673879} \qquad (3.3.15)$$

as

$\mathfrak{q} \neq 0$ (already proved)
we compute $\mathfrak{n}_{10}$,where

$$\alpha_1(\tau) = \frac{3202691933939614853691507761852340147907068571576934400000}{4309420939104419877798496448126324180384838309082007040} - \frac{5411165744893179925407529761055534224864000000}{1776733668740846607353511900279117159672 40}\tau + \frac{379578855212879782359435016542514466585404671941554083196926953057157120000000 0}{23654196171570624407054255546851376270303804452080109502434544276252645130240} - \frac{1972864985403134614012485924517056 0000}{6636530966361205395210860259556674}\tau^4 + \frac{72122286451423175623175623730611556583642087609917534976000000}{31419663699335273804006317105144804521396774714432 0}\tau^5 - \frac{1312464066858887802699417567800374864202273485063741440 0000}{10773552347761049694496241120315810450962095772705017 60}t^{14}$$

$$\beta_1(\tau) = -\frac{1}{2}\mathfrak{q} + \mathfrak{q}\boldsymbol{\tau}$$

$$\eta_{10} = -\frac{27395785168020478624216547254368302 9}{67217290298886375261175088099076092160000}\mathfrak{q}^9$$

which is constant number. Hence we get the multiplicity $\mu_{max} C_{14,1} \geq 10$ for class $C_{14,1}$.

Theorem (3.3.2)
Assume equation

$$\frac{d\mathbb{Z}}{d\tau} = \alpha_1(\tau)\mathbb{Z}^3 + \beta_1(\tau)\,\mathbb{Z}^2 \qquad (3.3.14)$$

where

$$\alpha_1(\tau) = \left(\frac{3202691933939614853691507761852340147907068571576934400000}{4309420939104419877798496448126324180384838309082007040} + \frac{3658607604596995788543904 0}{39981291695148615659468544 0}\in_1 - \frac{248364003462315031 76}{649348426300987400960}\in_2 + \frac{90704394}{39392864}\in_3 - \frac{933}{2156}\in_4 - \frac{1}{7}\in_5 - \frac{1}{8}\in_6 + \in_7\right) + \left(\frac{5411165744893179925407529761055534224864000000}{177673366874084660735351190027911715967240} - \frac{6181466264115}{16483910037640}\in_1 + \frac{1515921116556}{16483910037640}\in_1 + \in_2\right)\tau + \left(\frac{3795788552128797823594350165425144665854046719415540831969269530571571200000000000}{2365419617157062440705425554685137627030380445208010950243454427625264513024 00} - \frac{43361338363811790711956989240254767494225920000000}{2194553120508976317034536152502977985418509680640 0}\in_1 - \frac{6484492040811013152}{6658444684964151040}\in_2 - \frac{4277592}{403936}\in_3 + \in_4\right)t^2 + \left(-\frac{1972864985403134614012485924517056 0000}{6636530966361205395210860259556674} - \frac{225371}{615714}\in_1 + \in_2\right)t^4 + \left(\frac{72122286451423175623175623730611556583642087609917534976000000}{31419663699335273804006317105144804521396774714432 0} + \frac{8238917483003683444716 00}{2915005883801311705152 0}\in_1 - \frac{340437984750563700}{47343505000719680}\in_2 - \frac{224575}{28712}\in_3 + \frac{35}{66}\in_4 + \in_5\right)t^5 + \left(-\frac{1312464066858887802699417567800374864202273485063741440 0000}{10773552347761049694496241120315810450962095772705017 60} - \frac{1499298438568073087120070 40}{9995322923787153914867136 0}\in_1 + \frac{20613022375540122528}{162337106575246850240}\in_2 + \frac{13597688}{9848216}\in_3 + \frac{796}{1617}\in_4 - \frac{16}{35}\in_5 + \in_6\right)\tau^{14}$$

$$\beta_1(\tau) = -\frac{1}{2}\mathfrak{q} + \mathfrak{q}\tau + \in_8$$

with $\mathfrak{q} \neq 0$.
If $\in_i$, $1 \leq i \leq 8$ are selected to be non zero and every $\in_i$ is relatively small in comparison with $\in_{i-1}$, then the solution of $(3.3.14)$ will be eight non trivial real periodic solution.
Proof:

The coefficients are chosen such that origin multiplicity μ is 10 if $\in_i = 0$ for $1 \leq i \leq 8$. Select $\xi_1 \neq 0$ but $\in_i = 0$ for $2 \leq i \leq 8$, then we can check that $\eta_2 = \eta_3 = \cdots = \eta_6 = 0$ but η_9 is constant multiple of $\in_1$. Hence $\mu = 9$.So the multiplicity is decrease by one. Next alongside $\in_2 \neq 0$, by $\in_3 = \in_4 = \cdots = \in_8 = 0$, we have $\eta_2 = \eta_3 \ldots = \eta_7 = 0$ but $\eta_8 \neq 0$, and also η_8 is constant multiple of $\in_2$, so $\mu = 8$. Thus multiplicity is again decrease by one.
If $\in_2$ is little enough there will be two non trivial periodic solutions. By carry on this procedure, we will get eight real non trivial periodic solutions.

Corollary (3.3.3):
In (3.3.2) with given $\alpha_1(\tau)$and $\beta_1(\tau)$ the following equation

$$\frac{d\mathbb{z}}{d\tau} = \alpha_1(\tau)\mathbb{z}^3 + \beta_1(\tau)\mathbb{z}^2 + \Upsilon\mathbb{z} + \boldsymbol{\zeta}$$

(3.3.15)
have ten real periodic solutions if Type equation here. and $\boldsymbol{\zeta}$ are small enough.
Proof:
If $\Upsilon = 0$, $\boldsymbol{\zeta} = 0$ and $\mu = 2$ then the equation (3.3.15) has eight real periodic solutions. If γ is non zero then $\mu = 1$ and then by the same arguments used in above theorem we have nine real periodic solutions. Since $\mathbb{z} = 0$ is another solution therefore we have ten real periodic solutions.

3.6 Conclusion

In this thesis, intermittent arrangements are determined. The arrangements fulfilling z (β) = z (0), are called periodic solution of equation (1.1). A limit cycle is a periodic solution separated from all intermittent circles of the planar differential framework. We discovered occasional answer for arithmetical coefficients for classes $C_{11,4}$, $C_{11,5}$, $C_{11,6}$, $C_{17,1}$. We have most extreme variety 10 of class $C_{17,1}$ by utilizing traditional method. Since the arrangement of Hilbert's sixteenth issue is as yet distant, glancing through more breaking point cycles and heighten the overall upper limits could be a productive decision in moving toward the issue.

References

[1] A. Dosary, I.T. Khalil, “Limit cycles for planar differential systems with quasi-homogenous nonlinearities”. J Math Comp Sci. Vol. 9, pp. 365-371, (2019).
[2] A. Nawaz, “A Bifurcation of periodic solutions of certain classes of cubic nonautonomous differential equations”, (M.Phil. thesis). pp. 1-104, (2018).
[3] A.Gasull, J. Llibre, J. Sotomayer, “Limit cycles of vector field of the form X (v)= Av + f(v)Bv". J Differs Equat, Vol. 67, pp. 90-110, (1987).
[4] B. Van der pol, “On relaxation-oscillations”. Phil. Mag. Vol. 2, pp. 978-992, (1926).
[5] D.Hilbert,” Mathematical problems”. Bull Am Math Soc. Vol. 8, pp. 437-479, (1902).
[6] H. Poincare, “Memoire sur les courbes definies par une equation differentielles”. I, II. J Math Pures Appl. Vol. 7, pp. 375–442, (1881).
[7] H. Poincare, “Memoire sur les courbes definies par une equation differentilles”. I, II. J Math Pures Appl. Vol. 8, pp. 251-296, (1882).
[8] H. Poincare, “Sur les courbes definies par les equations differentielles”. III, IV. J Math Pures Appl. Vol. 1, pp. 167-224, (1885).
[9] H. Poincare, “Sur les courbes definies par les equations differentielles”. III, IV. J Math Pures Appl. Paris. Vol. 2, pp. 155-217, (1886).
[10] J. Guckenheimer, P. Holmes, “Nonlinear Oscillations Dynamical Systems and Bifurcations of Vector Fields”. 2nd ed. New York, NY: Springer-Verlag (1992).
[11] J. Huang, H. Liang, “A uniqueness criterion of limit cycles for planar polynomial system with homogenous nonlinearities”. J Maths Anal Appl. Vol. 457, pp. 498-521, (2018).

[12] J. Huang, H. Liang, J. Libre, “Non-existence and uniqueness of limit cycles for planar polynomial differential system with homogenous nonlinearities”. J Differ Equat. Vol. 265, pp. 888-913, (2018).

[13] L. Neto, “On the number of solutions of the equations $\frac{dx}{dt} = \sum_{j=0}^{n} a_j(t)t', 0 \leq t \leq 10 \leq$ t ≤ 1 for which $x\ (0)\ =\ x\ (1)$”. Invent Math. Vol. 59, pp. 67-76, (1980).

[14] M. Bohner, A. Gasull, C. Valls, “Periodic solution of linear, Riccati and Abel dynamic equations”. J Math Anals Appl. Vol. 470, pp.733-749, (2019).

[15] M.A.M Alwash, N.G. Llyod, “Non-autonomous equation related to polynomial two-dimensional system”. Proc R Soc Edinb. Vol. 105, pp. 129-152, (1987).

[16] M.A.M. Alwash, “Periodic solutions of polynomial non-autonomous differential equations”. Electron J Differ Equat. Vol. 84, pp. 1-8, (2005).

[17] M.A.M. Alwash, “Polynomial differential equations with small coefficients”. Discrete Contin Dyn Syst. Vol. 25, pp. 1129-1141, (2005).

[18] M.J. Alvarez, J.L. Bravo, M. Fernandez, R. Prohens, “Alien limit cycles in Abels equation”. J Math Anal Appl. Vol. 482, pp. 123-525, (2020).

[19] N. Hua, “The fixed point theory and the existence of the periodic solution on a nonlinear differential equation”. Js Appl Math. Vol.6, pp.725-989, (2018).

[20] N. Yasmin, “Bifurcating periodic solutions of polynomial system”. Punjab Univ J Math. pp. 43-46, (2001).

[21] N. Yasmin, “Closed orbits of certain two-dimensional cubic systems”, (Ph.D. thesis), University College of Wales Aberystwyth”, United Kingdom, pp.1-169, (1989).

[22] N. Yasmin, M. Ashraf, “Bifurcating periodic solutions of class $C_{2,4}$ and $C_{4,3}$ of research SCI”. BZU. pp. 14, (2003).

[23] N.G. Lloyd, “Small amplitude limit cycles of polynomial differential equations”. In: W.N. Everitt, R.T. Lewis, editors. Ordinary Differential Equations and Operators. Lecture Notes in Mathematics, (Berlin; Heidelberg: Springer), Vol. 1032, pp. 346-357, (1982).

[24] N.G. Llyod, “Limit cycles of certain polynomial systems”. In: Singh SP, editor. Non-linear Functional Analysis and Its Applications. NATO ASI Series, (C: Mathematical and Physical Sciences), Dordrecht: Springer, Vol. 173, pp. 317-326, (1986).

[25] N.G. Llyod, “The number of periodic solutions of the equation $\dot{z} = z^n + p_i(t)Z^{N-1} + \cdots + p_N(t)$”. Proc Lond Math Soc. Vol. 27, pp. 667-700, (1973).

[26] S. Akram, A. Nawaz, T. Abdeljawad, A. Ghaffar, K. S. Nisar, “Calculation of focal values for first order non-autonomous equation with algebraic and trigonometric coefficient”. Open physics, Vol. 18, pp. 738-750, (2020).

[27] S. Akram, A. Nawaz, H. Kalsoom, M. Idrees, Y. M. Chu, “Existence of multiple periodic solutions for cubic non- autonomous differential equation”. Mathematical problems in engineering, Vol. 8, pp. 1-14, (2020).

[28] S. Akram, A. Nawaz, N. Yasmin, H. Kalsoom, Y. M. Chu, “Periodic solutions for a first order non- autonomous differential equation with bifurcation analysis”. Journal of Taibah university for science, Vol. 14, pp. 1208-1217, (2020).

[29] S. Akram, “Periodic solutions of non-autonomous equations”, (M.Phil. thesis). pp. 1-40, (2007).

[30] S. Lynch, “Dynamical Systems with Applications Using MATLAB, Dynamical Systems with Applications Using Maple”. Boston, MA: Burkhouse (2004).

[31] Z. Zhou, Y. Yan, “On the composition center for a class of rigid system”. Bull Braz Math Soc. Vol. 51, pp. 139-155, (2020).