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Error Analysis and Accuracy Assessment of Runge-Kutta's Method in Solving Ordinary Differential Equations

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Abstract

The primary focus of this paper is the presentation of Runge-Kutta's method as a method for addressing initial value problems *(IVP)* in ordinary differential equations *(ODE).* This method demonstrates practical efficiency and suitability for addressing these problems. To ensure accuracy, we perform comparisons between numerical solutions and exact solutions. The numerical solutions are in good agreement with the exact solutions. To obtain greater solution accuracy the step size must be reduced to extremely small values. Our investigation reaches its terminus as we examine and calculate the discrepancies in Runge-Kutta's method across various step sizes.

Introduction

A differential equation represents a mathematical expression that incorporates independent variables alongside dependent variables and their respective derivatives concerning those independent variables. A differential equations order corresponds to the highest derivative present while its degree represents the power of this highest derivative once all fractional and negative exponents have been eliminated. An ordinary differential equation classifies as linear when it lacks any occurrence of the dependent variable multiplied by itself or its derivatives, whereas its presence renders the equation nonlinear. The solutions to the *nth* order equation depend on *n* parameters. Determining these parameters requires the provision of *n* conditions. The presence of these *n* conditions at a singular point defines the differential equation and its conditions as an *n*th order initial value problem. A problem becomes a boundary value problem *(BVP)* when *n* conditions are specified at multiple points. Science and engineering mathematical problems often rely on numerical methods because obtaining exact solutions proves difficult or impossible. The set of differential equations that permit analytical solutions remains severely restricted. Numerous analytical techniques exist to determine solutions for ordinary differential equations. A vast array of ordinary differential equations defies closed-form solutions through established analytical techniques, necessitating the application of numerical methods to obtain approximate solutions under specified initial conditions. A diverse array of practical numerical methods exists to address initial value problems in ordinary differential equations. This document introduces Runge-Kutta's Method as a technique to address initial value problems in ordinary differential equations. The examination of existing studies reveals numerous investigations into numerical solutions of initial value problems through the Runge-Kutta's method have been performed. Numerous authors have explored various methods including Runge-Kutta's method to achieve rapid high-accuracy solutions for *IVP*. The works by *[1]-[13]* explored numerical solutions for initial value problems in ordinary differential equations through Runge-Kutta's method alongside several other numerical techniques.

Runge-Kutta's Method

A complex set of iterative technique known as the Runge-Kutta's methods exits to represent *ODEs*. Numerical analysts favor their application because these methods deliver both exceptional performance and precise results. Among the methods in this family the RK_4 method stands out as the most frequently utilized member and is commonly known as RK_4. Runge-Kutta's method approximate the solution of an *IVP* of the form

$$\frac{dy}{dx} = f(x, y)$$

$$y\,(x_0) = y_0$$

Where y is unknown function of x. The RK_4 method is one of the most popular and widely used numerical methods for solving *ODEs*. It provides good valance between computational efforts and accuracy. *Compute the slopes.*

$$k_1 = f\,(x_n, y_n)$$

$$k_2 = f\,(x_n + \frac{h}{2}, y_n + \frac{h}{2}\,k_1)$$

$k_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right)$

$k_4 = f(x_n + h, y_n + hk_3)$

So $y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Use y_{n+1} as the new initial value and repeat the process for the next time step.

Now we use Runge-Kutta's method to numerically integrate equation is

$\frac{dy}{dx} = -y + e^{-x}$ $\quad y(0) = 1$

It is a non homogeneous first order linear *ODE* where $x=0$ to $x=2$ with a step size $h=0.5$

The initial condition at $x=0$ is $y=1$

The exact solution is $y=e^{-x}$

The true solution and Runge-Kutta's method solution at $x=0.5$ is given by *0.60653* and *0.60680*

$$Error\ \% = \left(\frac{E-T}{E}\right) X\ 100\ \%$$

$$= \left(\frac{0.60680-0.60653}{0.60680}\right) X100$$

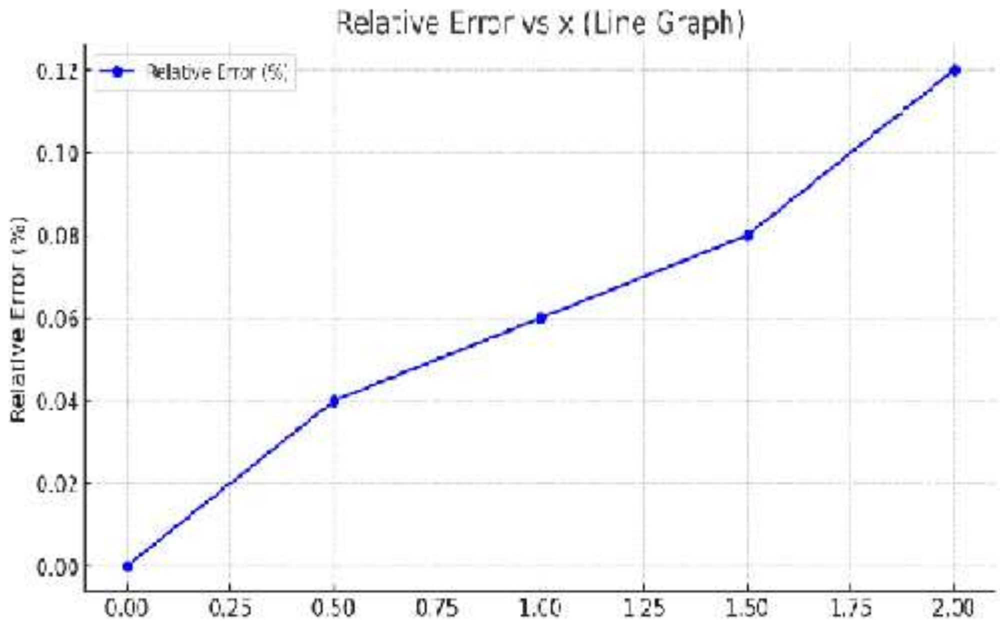
$$= 0.04\ \%$$

For 2^{nd} step the true solution and solution by Runge-Kutta's method at $x=0.5$ has been find out. The computation is repeated and the results are compiled in table as shown below.

x_i	Y_{true}	$Y_{Rungekutta}$	**% Relative Error**
0.0	1.00000	1.000000	0.0 %
0.5	0.60653	0.60680	0.04 %
1.0	0.36788	0.36810	0.06 %
1.5	0.22313	0.22330	0.08 %
2.0	0.13534	0.13550	0.12 %

Table

Comparison of y_{true} and $y_{R.K}$ values of integral of $y' = -y + e^{-x}$ with initial conditions that $y=1$ *at* $x=0$.the approximate values were computed using Runge-Kutta's method with a step size of $h = 0.5$



Error Analysis of Runge-Kutta's method

Initial proximity results in more accurate approximations which become more precise as distance increases. Somehow greater accuracy emerges when the points used in approximation become more densely packed together. Apparently, the approximation exceeds the actual curve for concave down functions while it falls below the actual curve for concave up functions. Runge-Kutta's method especially RK_4 is highly accurate numerical method for solving *ODEs* normally. The dependence of RK_4 accuracy on both the specific *ODE* characteristics and its precise solution emerges in complex ways. When the exact solution of an *ODE* is a polynomial of degree n and the RK_4 method used has degree m where $m \geq n$, results typically show no errors. The RK_4 method typically produces precise outcomes when applied to polynomials with degrees up to four. When the exact solution represents an exponential function such as e^{-x} the RK_4 method produces highly precise results despite not being entirely error free because of its inherent qualities. The Runge-Kutta's method shows exceptional performance with its relative errors staying well within acceptable boundaries despite minor increases. The method demonstrates exceptional accuracy in replicating true solution behavior throughout the interval which makes it a prevalent choice for ordinary differential equation solving. To address higher precision problems, one might consider employing refinement strategies or higher-order methods. Through the inclusion of additional terms in their approximations, higher-order Runge-Kutta's methods achieve enhanced accuracy. Adaptive step size techniques enable methods to modify step size dynamically using local error estimates to achieve a balance between accuracy and computational efficiency. The Runge-Kutta's method stands as a dependable and sturdy numerical instrument for ordinary differential equation solutions where its minimal relative errors demonstrate both precision and effectiveness. The process of finding numerical solutions for ordinary differential equations introduces two distinct error types: truncation error and round-off error. Truncation error emerges in numerical analysis when an infinite sum gets truncated and represented as a finite sum. A computer's restricted capacity to maintain significant digits leads to round-off errors. Truncation errors consist of two distinct components: one being the local truncation error which arises from method application during a single step, and the other being the propagated truncation error which emerges from approximations generated in preceding steps. The accuracy of calculations faces immediate disruption from local truncation error while propagated truncation error builds up and impacts results over extended periods. The combined truncation error represents the aggregate of these two distinct errors.

Conclusion

The accuracy of approximations tends to increase as you move away from initial close proximity. The proximity of approximation points paradoxically enhances accuracy. When dealing with concave down functions your approximation exceeds the actual curve while for concave up functions it falls short of the actual curve. The Runge-Kutta's method including RK_4 stands as an exceptionally precise numerical approach for solving ordinary differential equations under standard conditions. The reliability of RK_4 method strangely relies on the specific characteristics of the *ODE* and its precise solution. When the exact solution of an *ODE* is a polynomial with degree n and the RK_4 method applied has a degree m where $m \geq n$, we typically obtain results without errors. The RK_4 method typically delivers precise outcomes when applied to polynomials of degree *4* or lower. When dealing with an exact solution such as the exponential function $e^{-x}$the *RK4* method fails to achieve absolute error elimination yet delivers exceptionally precise outcomes because of its ostensibly high order approximation.

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